2010 Extension 1 Solution

Question 1

(a) $\int \frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\frac{x}{2}+C$

(b) $-2 \le x \le 2$

(c) $\ln(x+6) = 2\ln x = \ln x^2$

$x^2 - x - 6 = 0.$

$(x-3)(x+2) = 0.$

$\therefore x = 3, -2.$ But $x > 0, \therefore x = 3.$

(d) $\frac{3}{x+2} < 4$

$3(x+2) < 4(x+2)^2$

$(x+2)(3-4(x+2) < 0$

$(x+2)(-4x-5) < 0$

$x < -2$ or $x > -\frac{5}{4}.$

(e) Let $u = 1-x, du = -dx.$

When $x = 0, u = 1;$ when $x = 1, u = 0.$

$\int_0^1 x\sqrt{1-x}dx = \int_1^0 (1-u)\sqrt{u}(-du)$

$= \int_0^1 \left(\sqrt{u} - \sqrt{u^3}\right)du$

$= \left[\frac{2\sqrt{u^3}}{3} - \frac{2\sqrt{u^5}}{5}\right]_0^1$

$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.$

(f) $\Pr(x=2) = {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = \frac{625}{3888}.$

(b) (i) $\frac{dM}{dt} = 35.5ke^{-kt},$

but $35.5e^{-kt} = 36 - M,$

$\therefore \frac{dM}{dt} = k(36-M).$

(ii) $t = 10, M = 20,$

$20 = 36 - 35.5e^{-10k}$

$e^{-10k} = \frac{16}{35.5} = \frac{32}{71}.$

$-10k = \ln\frac{32}{71}.$

$\therefore k = -\frac{1}{10}\ln\frac{32}{71} = 0.080.$

(iii) As $t \to \infty, e^{-kt} \to 0, \therefore M \to 36$ tonnes.

(c) (i) $P(3) = 0 \Rightarrow 0 = 3a + b$

$P(-1) = 8 \Rightarrow 8 = -a + b$

$\therefore a = -2, b = 6.$

(ii) The remainder is $-2x+6.$

(d) $x^2 = r^2 - 36$

$2x\frac{dx}{dt} = 2r\frac{dr}{dt}.$

$x\frac{dx}{dt} = r\frac{dr}{dt}.$

$100x = \sqrt{x^2+36}\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{100x}{\sqrt{x^2+36}}$ km/h.

Question 2

(a) $f(x) = \int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}dx$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C.$

When $x = 0, f(0) = 2, \therefore 2 = C.$

$\therefore f(x) = \frac{x}{2} - \frac{\sin 2x}{4} + 2.$

Question 3

(a) (i) $\frac{5!}{2!} = 60.$

(ii) $4! = 24.$

(b) (i) $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$f''(x) = -2\left(e^{-x^2} - 2x^2e^{-x^2}\right) = -2\left(1-2x^2\right)e^{-x^2}.$

$f''(x) = 0$ when $2x^2 = 1, \therefore x = \pm\frac{1}{\sqrt{2}}.$

The points of inflexion has x-coordinates $\pm\frac{1}{\sqrt{2}}.$

(ii) $f(x)$ does not satisfy the horizontal line test.

(iii) $f: \ y = e^{-x^2}, x \ge 0, 0 < y \le 1.$

$f^{-1}: \quad x = e^{-y^2}$

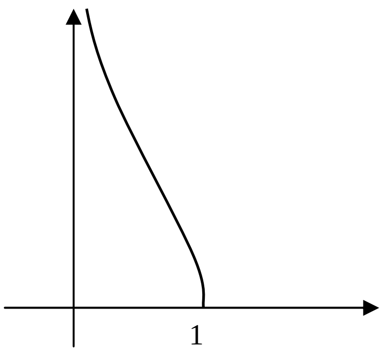
$-y^2 = \ln x$

$y^2 = -\ln x = \ln \frac{1}{x}.$

$\therefore y = +\sqrt{\ln \frac{1}{x}}$, since $y \ge 0$.

(iv) $0 < x \le 1$.

(v)



(vi) (1) Let $g(x) = x - e^{-x^2}$,

When $x = 0.6, g(0.6) = -0.098$

When $x = 0.7, g(0.7) = 0.087$

As $g(x)$ is a continuous function, and it changes signs between $x = 0.6$ and 0.7, it must have a zero between them.

(2) $x_0 = \frac{0.6+0.7}{2} = 0.65, g(0.65) = -0.005$

$x_1 = \frac{0.65+0.7}{2} = 0.675.$

As 0.65 and 0.675 agree to 1 decimal place, $\therefore$ to 1 decimal place, the solution is 0.7.

Question 4

(a) (i) $v^2 = -2\left(x^2 + 4x - 12\right) = -2(x+6)(x-2)$

$v^2 = 0$ when $x = -6, 2$.

(ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(-x^2 - 4x + 12\right) = -2x - 4.$

(iii) Maximum speed occurs when $a = 0, \therefore x = -2$.

$\therefore v^2 = 32, \therefore$ Maximum speed $= \sqrt{32} = 4\sqrt{2}.$

(b) (i) $2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right)$

$= 2\cos\theta + 2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) = 3\cos\theta - \sqrt{3}\sin\theta$

$= \sqrt{3}\left(\sqrt{3}\cos\theta - \sin\theta\right) = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$

$= 2\sqrt{3}\left(\cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}\right)$

$= 2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right).$

(ii) $2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) = 3$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\theta + \frac{\pi}{6} = \pm\frac{\pi}{6} + k2\pi, k \in J.$

$\therefore \theta = k2\pi, -\frac{\pi}{3} + k2\pi.$

For $0 < \theta < 2\pi, \theta = \frac{5\pi}{3}.$

(c) By definition, $SP = PM = a + ap^2$.

From $y = px - ap^2$, let $x = 0, y = -ap^2, \therefore L(0, -ap^2)$

$\therefore SL = a - (-ap^2) = a + ap^2.$

$M(2ap, -a), \therefore LM = \sqrt{(2ap-0)^2 + (-a+ap^2)^2}$

$= \sqrt{4a^2p^2 + a^2 - 2a^2p^2 + a^2p^4}$

$= \sqrt{a^2 + 2a^2p^2 + a^2p^4} = \sqrt{(a+ap^2)^2} = a + ap^2.$

$\therefore SLMP$ is a rhombus (4 sides equal)

Question 5

(a) (i) $\tan 20° = \frac{PL}{AP} = \frac{1}{AP}, \therefore AP = \frac{1}{\tan 20°}.$

$\tan 3° = \frac{PT}{AP} = PT\tan 20°. \qquad (1)$

$\tan 30° = \frac{PT}{BP}, \therefore \frac{1}{\sqrt{3}} = \frac{PT}{BP}. \qquad (2)$

$\frac{(1)}{(2)} \Rightarrow \sqrt{3}\tan 3° = BP\tan 20°.$

$\therefore BP = \frac{\sqrt{3}\tan 3°}{\tan 20°}.$

(ii) $AB = AP - BP = \frac{1-\sqrt{3}\tan 3°}{\tan 20°}.$

(b) (i) $f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times \frac{-1}{x^2}$

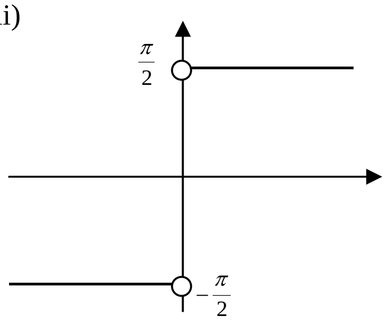
$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0.$

$\therefore f(x)$ is constant.

Substitute $x = 1$, $f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

$\therefore$ For $x > 0$, $f(x) = \frac{\pi}{2}$.

(ii)



(c) (i) Exterior angle in a Δ is the sum of the two opposite interior angles.

(ii) The angle between a tangent and a chord is equal to any angle in the alternate segment.

(iii) $\angle ABD = \angle TAC$ (same theorem in (ii)),

$\therefore \angle XDB = \angle CAD$ (from (i) and (ii)),

But $\angle XAD = \angle XDB$ (same theorem in (ii)),

$\therefore \angle XAD = \angle CAD.$

$\therefore AD$ bisects $\angle BAC$.

Question 6

(a) (i) $\cos(A-B) = \cos A\cos B + \sin A \sin B$

$= \cos A \cos B\left(1+\frac{\sin A \sin B}{\cos A \cos B}\right)$

$= \cos A \cos B\left(1 + \tan A \tan B\right)$

(ii) If $\tan A \tan B = -1, \cos(A-B) = 0$

$\therefore A - B = \pm\frac{\pi}{2} + k2\pi.$

But $0 < B < \frac{\pi}{2}$ and $B < A < \pi, \therefore A - B = \frac{\pi}{2}$.

(b) (i) Substitute $x = d, y = h$,

$d = vt\cos\theta, \therefore t = \frac{d}{v\cos\theta}.$

$h = vt\sin\theta - 5t^2 = v\frac{d}{v\cos\theta}\sin\theta - 5\frac{d^2}{v^2\cos^2\theta}$

$= d\tan\theta - \frac{5d^2}{v^2\cos^2\theta}.$

$\therefore \frac{5d^2}{v^2\cos^2\theta} = d\tan\theta - h,$

$v^2\cos^2\theta = \frac{5d^2}{d\tan\theta - h}.$

But $h = d\tan\alpha$,

$\therefore v^2\cos^2\theta = \frac{5d^2}{d\tan\theta - d\tan\alpha} = \frac{5d}{\tan\theta - \tan\alpha}$

$v^2 = \frac{5d}{\cos^2\theta\tan\theta - \cos^2\theta\tan\alpha}$

$= \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha}.$

(ii) (1) When $\theta \to \alpha, v^2 \to \frac{5d}{\cos\alpha\sin\alpha - \cos\alpha\sin\alpha} \to \frac{5d}{0}$

$\to \infty, \therefore v \to \infty.$

(2) When $\theta \to \frac{\pi}{2}, \cos\frac{\pi}{2} \to 0, v^2 \to \frac{5d}{0} \to \infty, \therefore v \to \infty.$

(iii) $F'(\theta) = -\sin^2\theta + \cos^2\theta + 2\cos\theta\sin\theta\tan\alpha$

$= \cos 2\theta + \sin 2\theta \tan\alpha$

$= 0$ when $\cos 2\theta = -\sin 2\theta\tan\alpha$

$\therefore \tan 2\theta \tan\alpha = -1.$

(iv) From (a) (ii), $2\theta - \alpha = \frac{\pi}{2}, \therefore \theta = \frac{\alpha}{2} + \frac{\pi}{4}.$

(v) $v^2 = \frac{5d}{F(\theta)}, \therefore \frac{d}{d\theta}\left(v^2\right) = \frac{-5d \times F'(\theta)}{F^2(\theta)}.$

$\frac{d}{d\theta}\left(v^2\right) = 0$ when $F'(\theta) = 0$, i.e. $\theta = \frac{\alpha}{2} + \frac{\pi}{4}.$

As v^2 is a continuous function for $0 < \theta < \frac{\pi}{2}$, and v^2 $\to \infty$ for $x \to 0$ and $x \to \frac{\pi}{2}, v^2$ has a minimum value.

Further, as $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$ is the unique value for which $F'(\theta) = 0$, $0 < \theta < \frac{\pi}{2}, v^2$ is minimum when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$

Question 7

(a) Let $n=1, 47+53\times147^0=47+53=100$, which is divisible by 100.

Assume $47^n+53\times147^{n-1}$ is divisible by 100,

let $47^n+53\times147^{n-1}=100M$, where M is an integer.

$\therefore 47^n=100M-53\times147^{n-1}$.

$$\begin{aligned}47^{n+1}+53\times147^n &= 47\left(100M-53\times147^{n-1}\right)+53\times147^n\\ &= 4700M+53\times147^{n-1}(-47+147)\\ &= 4700M+5300\times147^{n-1}\\ &= 100\left(47M+53\times147^{n-1}\right).\end{aligned}$$

$\therefore 47^{n+1}+53\times147^n$ is divisible by 100.

By the Principle of Induction, $47^n+53\times147^{n-1}$ is divisible by 100 for all $n\ge1$.

(b) (i) Put $x=1$, $2^n=\sum_{k=0}^{n}\binom{n}{k}$.

(ii) Put $n=100$, $2^{100}=\sum_{k=0}^{100}\binom{100}{k}$.

(iii) Differentiating both sides of

$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{0}x^n$ gives

$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+\ldots+n\binom{n}{0}x^{n-1}$.

Put $x=1$, $n2^{n-1}=k\sum_{k=1}^{n}\binom{n}{k}$.

(c) (i) Since there are two colours, selecting 1 ball has 2 ways (either R or B), selecting 2 balls has 3 ways (2R, 1R1B, 2B), selecting 3 balls has 4 ways (3R, 1R2B, 2R1B, 3B), and so on.

$\therefore$ The number of ways of selecting r balls is the number of terms of $\binom{r}{0},\binom{r}{1},\binom{r}{2},\ldots,\binom{r}{r}$, i.e. $r+1$ ways.

(ii) Selecting $n-r$ balls from n different balls has $\binom{n}{n-r}$ ways. But $\binom{n}{n-r}=\binom{n}{r}$, due to symmetry.

(iii) In part (c) (ii), the number of ways of selecting $(n-r)$ balls from n different balls is $\binom{n}{r}$, and in part (i), the number of ways of selecting r balls from n identical red balls and n identical blue balls is $r+1$.

$\therefore$ The total number of ways of selecting $n, n-1,\ldots,0$ white balls is $\binom{n}{0}+2\binom{n}{1}+3\binom{n}{2}+\ldots+(n+1)\binom{n}{n}$, given that once $n-r$ white balls are chosen, there is $r+1$ possible number of ways of choosing r balls from the identical red or blue balls.

As $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$, from part (b) (i),

$\binom{n}{1}+2\binom{n}{2}+\ldots+n\binom{n}{n}=n2^{n-1}$, from part (b) (iii),

$\therefore$ the total $=2^{n-1}+n2^{-1}=(n+2)2^{n-1}$.

Note: The question was worded poorly. What they meant was n identical red balls, n identical blue balls and n labelled white balls.