

**Question 1**

$$(a) x = \frac{1 \times -1 + 4 \times 9}{1 + 4} = \frac{35}{5} = 7$$

$$y = \frac{1 \times -2 + 4 \times 3}{1 + 4} = \frac{10}{5} = 2, \therefore P(7, 2)$$

$$(b) y = \frac{\sin^2 x}{x}$$

$$\frac{dy}{dx} = \frac{2x \sin x \cos x - \sin^2 x}{x^2} = \frac{\sin x (2x \cos x - \sin x)}{x^2}$$

$$(c) \frac{4-x}{x} < 1$$

$$(4-x)x < x^2$$

$$4x - 2x^2 < 0$$

$$2x(2-x) < 0$$

$$\therefore x < 0 \text{ or } x > 2$$

$$(d) u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx.$$

When  $x = 1, u = 1$ ; when  $x = 4, u = 2$ .

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int_1^2 e^u du = 2 [e^u]_1^2 = 2(e^2 - e).$$

$$(e) \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2} = \frac{2\pi}{3}.$$

$$(f) x^2 + e \geq e$$

$$\therefore \ln(x^2 + e) \geq \ln e = 1.$$

$$\therefore \text{Range: } y \geq 1.$$

**Question 2**

$$(a) P(3) = 12, \therefore 27 - 9a + 3 = 12, \therefore a = 2$$

$$\therefore P(-1) = -1 - 2 - 1 = -4$$

$\therefore$  Remainder is  $-4$ .

$$(b) f(x) = \cos 2x - x, \therefore f\left(\frac{1}{2}\right) = 0.0403$$

$$f'(x) = -2 \sin 2x - 1, \therefore f'\left(\frac{1}{2}\right) = -2.6829$$

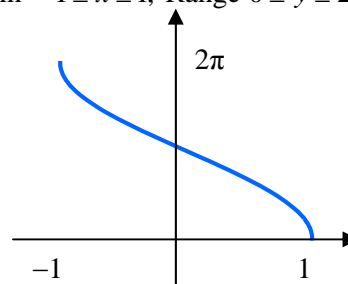
$$x_2 = \frac{1}{2} - \frac{0.0403}{-2.6829} = 0.5150 = 0.52 \text{ (2 dp)}$$

$$(c) \left(3x - \frac{4}{x}\right)^8 = \frac{(3x^2 - 4)^8}{x^8}.$$

$\therefore$  Coefficient of  $x^2$  = coefficient of  $x^{10}$  in the expansion of  $(3x^2 - 4)^8 = {}^8C_5 3^5 (-4)^3 = -870912$ .

$$(d) f(x) = 2 \cos^{-1} x.$$

Domain  $-1 \leq x \leq 1$ , Range  $0 \leq y \leq 2\pi$ .



$$(e) (i) 40!$$

$$(ii) 3!37!$$

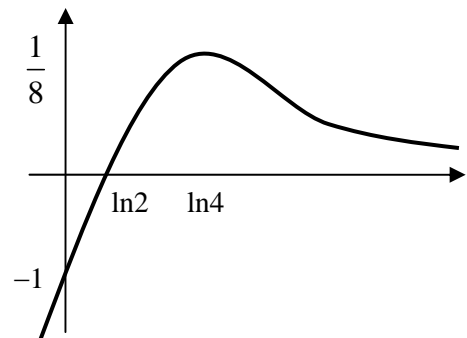
### Question 3

- (a) (i)  $\frac{dx}{dt} = -nA \sin nt + nB \cos nt$   
 $\frac{d^2x}{dt^2} = -n^2 A \cos nt - n^2 B \sin nt$   
 $= -n^2 (A \cos nt + B \sin nt) = -n^2 x$   
 (ii)  $t = 0, x = 0, \therefore 0 = A \cos n \cdot 0, \therefore A = 0$   
 $t = 0, \frac{dx}{dt} = 2n, \therefore 2n = nB \cos 0, \therefore B = 2$   
 (iii) Period =  $\frac{2\pi}{n}$ ,  $\therefore$  The time taken to travel from the origin to its max. distance is  $\frac{1}{4} \times$  period  
 $= \frac{\pi}{2n}$ .  
 (iv) Total distance for 1 period =  $4 \times$  amplitude  
 $= 4 \times 2 = 8$

- (b) (i)  $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1} = 2t$   
 $y - t^2 = 2t(x - t)$   
 $y = 2tx - 2t^2 + t^2$   
 $\therefore y = 2tx - t^2$ . (1)  
 (ii)  $y = 2(1-t)x - (1-t)^2$  (2)  
 (iii) (1) = (2) gives  $2tx - t^2 = 2(1-t)x - (1-t)^2$   
 $2tx - t^2 = 2x - 2tx - 1 + 2t - t^2$   
 $4tx - 2x = -1 + 2t$   
 $2x(2t - 1) = -1 + 2t$   
 $2x = 1$   
 $\therefore x = \frac{1}{2}$   
 Sub.  $x = \frac{1}{2}$  to (1) gives  $y = t - t^2$ .  
 $\therefore R\left(\frac{1}{2}, t - t^2\right)$   
 (iv) The locus of  $R$  is the line  $x = \frac{1}{2}$ .  
 But  $y = t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \leq \frac{1}{4}$ .  
 Since  $P \neq Q$ , i.e.  $t \neq \frac{1}{2}, \therefore y < \frac{1}{4}$ .

### Question 4

- (a) (i)  $f'(x) = -e^{-x} + 4e^{-2x}$   
 (ii)  $f'(x) = 0$  gives  $e^{-x} = 4e^{-2x}$   
 $1 = 4e^{-x}$   
 $e^x = 4$   
 $x = \ln 4$ .  
 $y = e^{-\ln 4} - 2e^{-2\ln 4} = \frac{1}{4} - 2 \times \frac{1}{16} = \frac{1}{8}$ .  
 $\therefore \text{SP} \left( \ln 4, \frac{1}{8} \right)$ .  
 Note: The question does not ask you to prove that this is maximum.  
 (iii)  $f(\ln 2) = e^{-\ln 2} - 2e^{-2\ln 2} = \frac{1}{2} - 2 \times \frac{1}{4} = 0$ .  
 (iv) As  $x \rightarrow +\infty, e^{-x} \rightarrow 0, e^{-2x} \rightarrow 0, \therefore y \rightarrow 0$   
 As  $x \rightarrow -\infty, y \approx -2e^{-2x} \rightarrow -\infty$   
 (v) When  $x = 0, y = 1 - 2 = -1$ .  
 (vi)



- (b) (i) Angle at the centre is twice angle on the circumference subtending the same arc.  
 (ii)  $\angle ADC = 2x$  (exterior angle in a  $\Delta$  = sum of two opposite interior angles)  
 $\therefore \angle ADC = \angle AOC$ .  
 $\therefore ACDO$  is cyclic (equal angles subtending the same arc)  
 (iii)  $OC = OA =$  radius of circle  $ABC$ .  
 $PA = PC =$  radius of circle  $ACDO$ .  
 $\therefore$  Both  $O$  and  $P$  belong to the perpendicular bisector of  $AC$ .  
 $\therefore O, P$  and  $M$  are collinear.

**Question 5**

$$(a) (i) \frac{SP}{TQ} = \frac{SN}{TN} \text{ (corresponding sides of similar } \Delta s)$$

$$\text{But } SN = 2, \cos \theta = TQ, \text{ and } \sin \theta = OT \\ = 1 - TN, \therefore TN = 1 - \sin \theta.$$

$$\therefore \frac{SP}{\cos \theta} = \frac{2}{1 - \sin \theta}.$$

$$\therefore SP = \frac{2 \cos \theta}{1 - \sin \theta}.$$

$$(ii) \frac{\cos \theta}{1 - \sin \theta} = \frac{\cos \theta}{1 - \sin \theta} \times \frac{1 + \sin \theta}{1 + \sin \theta}$$

$$= \frac{\cos \theta + \cos \theta \sin \theta}{\cos^2 \theta}$$

$$= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} = \sec \theta + \tan \theta.$$

$$(iii) \angle SNP = \frac{1}{2} \angle SOP \text{ (angle at the circumference}$$

is half angle at the centre subtending the same arc)

$$= \frac{1}{2} \left( \frac{\pi}{2} + \theta \right) = \frac{\pi}{4} + \frac{\theta}{2}.$$

$$(iv) \tan \angle SNP = \frac{TQ}{TN} = \frac{\cos \theta}{1 - \sin \theta}$$

$$\therefore \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right) = \sec \theta + \tan \theta.$$

$$(v) \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right) = \sqrt{3}$$

$$\frac{\pi}{4} + \frac{\theta}{2} = \frac{\pi}{3} + k\pi$$

$$\frac{\theta}{2} = \frac{\pi}{12} + k\pi$$

$$\therefore \theta = \frac{\pi}{6} + 2k\pi.$$

$$\therefore \text{For } -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \theta = \frac{\pi}{6}.$$

$$(b) (i) t = 1, T = 20, \therefore 20 = 5 + 25e^{-k}.$$

$$e^{-k} = \frac{15}{25} = \frac{3}{5}.$$

$$-k = \ln \frac{3}{5}$$

$$\therefore k = \ln \frac{5}{3}.$$

$$(ii) t = 0, T = 30, A = 22, \therefore 30 = 22 + B$$

$$\therefore B = 8.$$

$$\therefore T = 22 + 8e^{\left(-\ln \frac{5}{3}\right)t}$$

$$\therefore \text{When } T = 37, 37 = 22 + 8e^{\left(-\ln \frac{5}{3}\right)t}$$

$$\frac{15}{8} = e^{\left(-\ln \frac{5}{3}\right)t}$$

$$\ln \frac{15}{8} = -\ln \frac{5}{3} t$$

$$t = \frac{\ln \frac{15}{8}}{-\ln \frac{5}{3}} = -1.23 = -1 \text{ hour } 14 \text{ minutes ago.}$$

$\therefore$  The object had a temperature of  $37^\circ$  at 08:46.

**Question 6**

(a) (i) Let  $n = 1$ ,  $LHS = 1 \times 5 = 5$ ,

$$RHS = \frac{1}{6} \times 1 \times 2 \times 15 = 5. \therefore \text{True.}$$

Assume  $1 \times 5 + 2 \times 6 + \dots + n(n+4)$

$$= \frac{1}{6}n(n+1)(2n+13).$$

RTP  $1 \times 5 + 2 \times 6 + \dots + (n+1)(n+5)$

$$= \frac{1}{6}(n+1)(n+2)(2n+15).$$

$$LHS = \frac{1}{6}n(n+1)(2n+13) + (n+1)(n+5)$$

$$= \frac{1}{6}(n+1)(n(2n+13) + 6(n+5))$$

$$= \frac{1}{6}(n+1)(2n^2 + 13n + 6n + 30)$$

$$= \frac{1}{6}(n+1)(2n^2 + 19n + 30)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+15) = RHS.$$

$\therefore$  True for  $n + 1$ .

$\therefore$  True for all  $n \geq 1$  by the principle of Induction.

(b) (i) The ball strikes the ground when  $y = 0$ ,

$$\therefore h = \frac{1}{2}gt^2$$

$$\therefore t = \sqrt{\frac{2h}{g}} \text{ s.}$$

(ii) At the moment of impact,  $\tan(-45^\circ) = \frac{\dot{y}}{\dot{x}}$

$$\therefore -1 = \frac{-gt}{v}$$

$$\therefore v = gt = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}.$$

$$\therefore x = \sqrt{2gh}\sqrt{\frac{2h}{g}} = 2h.$$

$$\therefore d = 2h.$$

(c) (i)  $\Pr(\text{hits target at least once}) = 1 - \Pr(\text{hits no target}) = 1 - (1-p)^2 = 2p - p^2$ .

Alternatively,  $\Pr = \Pr(\text{hits target once}) + \Pr(\text{hits target twice}) = {}^2C_1 p(1-p) + p^2 = 2p - p^2$ .

(ii)  $\Pr(\text{hits target at least twice}) = \Pr(\text{hits target twice}) + \Pr(\text{hits target 3 times})$

$$= {}^3C_2 p^2(1-p) + p^3$$

$$= 3p^2 - 3p^3 + p^3 = 3p^2 - 2p^3$$

(iii)  $2p - p^2 - (3p^2 - 2p^3) = 2p - 4p^2 + 2p^3$

$$= p(p^2 - 2p + 2) = p((p+1)^2 + 1) > 0,$$

$\therefore \Pr(\text{win game 1}) > \Pr(\text{win game 2})$

(iv)  $2p - p^2 = 2(3p^2 - 2p^3)$ .

$$2p - p^2 = 6p^2 - 4p^3$$

$$4p^3 - 7p^2 + 2p = 0$$

$$p(4p^2 - 7p + 2) = 0$$

$$\therefore p = \frac{7 \pm \sqrt{17}}{8}.$$

As  $0 < p < 1$ ,  $p = \frac{7 - \sqrt{17}}{8} \approx 0.36$ .

**Question 7**

(a) (i) Volume remained = Volume of the cone of radius  $h$ , height  $h$  - Volume of the cone of radius  $\ell$ , height  $\ell$

$$= \frac{1}{3}\pi h^2 \times h - \frac{1}{3}\pi \ell^2 \times \ell$$

$$= \frac{\pi}{3}(h^3 - \ell^3)$$

$$(ii) \frac{dV}{dt} = \frac{\pi}{3} \times -3\ell^2 \times \frac{d\ell}{dt}$$

$$= -\pi \ell^2 \times \frac{d\ell}{dt}$$

$$= -\pi \times 4 \times 10 = -40\pi \text{ cm}^3/\text{s}.$$

(iii) When the lower cone has lost  $\frac{1}{8}$  of its water,

$$V = \frac{7}{8} \times \frac{\pi}{3} h^3.$$

$$\frac{7}{8} \times \frac{\pi}{3} h^3 = \frac{\pi}{3} (h^3 - \ell^3)$$

$$h^3 = 8\ell^3$$

$$\therefore h = 2\ell \text{ or } \ell = \frac{h}{2}.$$

$$\therefore \frac{dV}{dt} = -\pi \ell^2 \times \frac{d\ell}{dt} = -\pi \frac{h^2}{4} \times 10 = -\frac{5\pi h^2}{2} \text{ cm}^3/\text{s}.$$

$$(b) (i) (1+x)^n = \sum_{r=0}^n \binom{n}{r} x^r$$

Differentiating both sides wrt  $x$ ,

$$n(1+x)^{n-1} = \sum_{r=1}^n \binom{n}{r} r x^{r-1}.$$

Multiplying both sides by  $x$ ,

$$nx(1+x)^{n-1} = \sum_{r=1}^n \binom{n}{r} r x^r. \quad (1)$$

(ii) Differentiating (1) again,

$$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \sum_{r=1}^n \binom{n}{r} r^2 x^{r-1} \quad (2)$$

Let  $x = 1$

$$n2^{n-1} + n(n-1)2^{n-2} = \sum_{r=1}^n \binom{n}{r} r^2$$

$$\text{LHS} = 2^{n-2}(2n + n^2 - n) = 2^{n-2}n(n+1)$$

$$\therefore n(n+1)2^{n-2} = \sum_{r=1}^n \binom{n}{r} r^2. \quad (3)$$

(iii) In (2), let  $x = -1$ ,

$$0 = \sum_{r=1}^n \binom{n}{r} r^2 (-1)^{r-1}. \quad (4)$$

(3) - (4) gives

$$n(n+1)2^{n-2} = \sum_{r=1}^n \binom{n}{r} r^2 - \sum_{r=1}^n \binom{n}{r} r^2 (-1)^{r-1}.$$

$$\text{RHS} = \binom{n}{1} 1^2 + \binom{n}{2} 2^2 + \binom{n}{3} 3^2 + \dots + \binom{n}{n} n^2$$

$$- \binom{n}{1} 1^2 + \binom{n}{2} 2^2 - \binom{n}{3} 3^2 + \dots + (-1)^n \binom{n}{n} n^2,$$

The last term, due to  $n$  being even,  $= \binom{n}{n} n^2$ .

$$\therefore n(n+1)2^{n-2} = 2 \left[ \binom{n}{2} 2^2 + \binom{n}{4} 4^2 + \dots + \binom{n}{n} n^2 \right]$$

$$\therefore \binom{n}{2} 2^2 + \binom{n}{4} 4^2 + \dots + \binom{n}{n} n^2 = n(n+1)2^{n-3}.$$