

Multiple choice questions

Q1 (C), $P(2) = 8 - 16 - 12 + k = 0, \therefore k = 20$

Q2 (D)

Q3 (C), $= \pi - \frac{1}{2} \times \frac{3\pi}{5} = \frac{7\pi}{10}$

Q4 (D)

Q5 (A)

Q6 (B)

Q7 (A)

Q8 (D), $\sin 2\theta = 2 \sin \theta \cos \theta = 2 \times \frac{5}{13} \times \frac{12}{13} = \frac{120}{169}$

Q9 (B)

Q10 (C), The solution is $-2 \leq x \leq 3$

Question 11

(a) $\alpha\beta\gamma = -\frac{7}{2}$

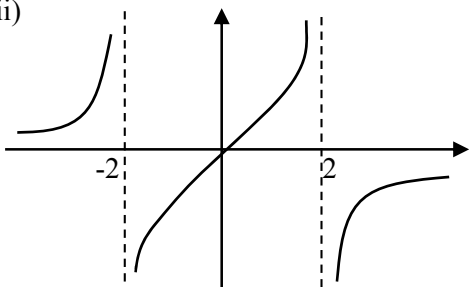
(b) $\int \frac{1}{\sqrt{49-4x^2}} dx = \frac{1}{2} \sin^{-1} \frac{2x}{7} + C$

(c) ${}^{10}C_7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^3$

(d) (i) $f'(x) = \frac{(4-x^2)+2x^2}{(4-x^2)^2} = \frac{4+x^2}{(4-x^2)^2} > 0$ since

$4+x^2 > 0$

(ii)



(e) $\lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{3x} = \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \times \frac{1}{6} = \frac{1}{6}$

(f) $\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x}+1} dx = \frac{1}{3} \left[\tan^{-1} e^{3x} \right]_0^{\frac{1}{3}}$
 $= \frac{1}{3} (\tan^{-1} e - \tan^{-1} 1) = \frac{1}{3} \left(\tan^{-1} e - \frac{\pi}{4} \right)$

(g) $\frac{d}{dx} (x^2 \sin^{-1} 5x) = 2x \sin^{-1} 5x + \frac{5x^2}{\sqrt{1-25x^2}}$

Question 12

(a) (i) $\sqrt{3} \cos x - \sin x = 2 \cos \left(x + \frac{\pi}{6} \right)$

(ii) $\sqrt{3} \cos x - \sin x = 1$

$2 \cos \left(x + \frac{\pi}{6} \right) = 1$

$\cos \left(x + \frac{\pi}{6} \right) = \frac{1}{2}$

$x + \frac{\pi}{6} = \pm \frac{\pi}{3} + k2\pi, k \in \mathbb{Z}$

$x = \frac{\pi}{6} + k2\pi, -\frac{\pi}{2} + k2\pi$

\therefore For $0 \leq x < 2\pi, x = \frac{\pi}{6}, \frac{3\pi}{2}$

(b) $V = \pi \int_0^{\frac{3\pi}{2}} 9 \sin^2 \frac{x}{2} dx = \frac{9\pi}{2} \int_0^{\frac{3\pi}{2}} (1 - \cos x) dx$

$= \frac{9\pi}{2} \left[x - \sin x \right]_0^{\frac{3\pi}{2}} = \frac{9\pi}{2} \left(\frac{3\pi}{2} + 1 \right) u^3$

(c) $T = A + Be^{-kt}$ where $A = 22^\circ$

When $t = 0, T = 80^\circ, \therefore 80^\circ = 22^\circ + B, \therefore B = 58^\circ$

When $t = 10, T = 60^\circ, \therefore 60^\circ = 22^\circ + 58^\circ e^{-10k}$

$\therefore e^{-10k} = \frac{38}{58} = \frac{19}{29}$

$\therefore k = -\frac{1}{10} \ln \frac{19}{29}$

$\therefore T = 22 + 58e^{\left(\frac{1}{10} \ln \frac{19}{29}\right)t}$

When $T = 40, 40 = 22 + 58e^{\left(\frac{1}{10} \ln \frac{19}{29}\right)t}$

$e^{\left(\frac{1}{10} \ln \frac{19}{29}\right)t} = \frac{18}{58} = \frac{9}{29}$

$\left(\frac{1}{10} \ln \frac{19}{29}\right)t = \ln \frac{9}{29}$

$\therefore t = 27.67 \approx 28$ minutes.

(d) (i) The equation of (ℓ) is $2x - y - 1 = 0$

$D(t) = \frac{|2t - (t^2 + 3) - 1|}{\sqrt{2^2 + 1^2}} = \frac{|-t^2 + 2t - 4|}{\sqrt{5}} = \frac{t^2 - 2t + 4}{\sqrt{5}}$

since $t^2 - 2t + 4 = (t-1)^2 + 3 > 0$.

(ii) $\frac{dD}{dt} = \frac{1}{\sqrt{5}}(2t - 2)$

$\frac{dD}{dt} = 0$ when $t = 1$

$$\frac{d^2D}{dt^2} = \frac{2}{\sqrt{5}} > 0, \therefore \text{minimum when } t = 1.$$

(iii) The gradient of the tangent at P is $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
 $= \frac{2t}{1}$. When $t = 1$, $m = 2$, which is the same gradient of (ℓ) . \therefore The tangent is parallel to (ℓ) .

(e) $v^2 = k - 9x^2$

$$\ddot{x} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{1}{2} \frac{d}{dx} (k - 9x^2) = -9x.$$

\therefore SHM, with $n = 3$

$$\text{Period} = \frac{2\pi}{n} = \frac{2\pi}{3}.$$

Question 13

(a) (i) $V = \frac{4}{3} \pi r^3, \therefore \frac{dV}{dr} = 4\pi r^2 = A$

$$\frac{dr}{dt} = \frac{dr}{dV} \frac{dV}{dt} = \frac{1}{A} \times -10^{-4} A = -10^{-4}$$

(ii) $r = -10^{-4}t + C$

When $t = 0, V = 10^{-6}, r = \sqrt[3]{\frac{3 \times 10^{-6}}{4\pi}}, \therefore C = \sqrt[3]{\frac{3 \times 10^{-6}}{4\pi}}$

$$\therefore r = -10^{-4}t + \sqrt[3]{\frac{3 \times 10^{-6}}{4\pi}}$$

When $r = 0, t = \frac{\sqrt[3]{\frac{3 \times 10^{-6}}{4\pi}}}{10^{-4}} = 62 \text{ s.}$

(b) (i) $x = \frac{2 \times ap - 0}{2 - 1} = 2ap,$

$$y = \frac{2 \times 0 - (2a + ap^2)}{2 - 1} = -2a - ap^2$$

(ii) $x = 2ap, \therefore p = \frac{x}{2a}.$

$$y = -2a - a \frac{x^2}{4a^2} = -2a - \frac{x^2}{4a}$$

$$-4ay - 8a^2 = x^2$$

$$-4a(y + 2a) = x^2$$

Focal length = a , Vertex $y = -2a$, Directrix $y = -a$

\therefore The same directrix and focal length as $4ay = x^2$.

(c) (i) The particle fired from A has equations

$$x = ut \cos \alpha, y = ut \sin \alpha - \frac{gt^2}{2}, \text{ if the origin}$$

is chosen at A .

$$\therefore \dot{y} = u \sin \alpha - gt$$

Maximum height is reached when $\dot{y} = 0$,

$$\therefore \text{when } t = \frac{u \sin \alpha}{g}.$$

(ii) The particle fired from B has equations

$$x = wt \cos \beta, y = wt \sin \beta - \frac{gt^2}{2}, \text{ if the origin}$$

is chosen at B .

$$\therefore \text{It reaches maximum height when } t = \frac{w \sin \beta}{g}.$$

\therefore When they both reach maximum height,

$$\frac{u \sin \alpha}{g} = \frac{w \sin \beta}{g}$$

$$\therefore u \sin \alpha = w \sin \beta$$

(iii) d is the distance travelled by both particles,

$$d = ut \cos \alpha + wt \cos \beta$$

$$= \frac{u^2 \sin \alpha \cos \alpha}{g} + \frac{w^2 \sin \beta \cos \beta}{g}$$

$$= \frac{u \cos \alpha}{g} w \sin \beta + \frac{w \cos \beta}{g} u \sin \alpha$$

$$= \frac{uw}{g} (\cos \alpha \sin \beta + \sin \alpha \cos \beta)$$

$$= \frac{uw}{g} \sin(\alpha + \beta)$$

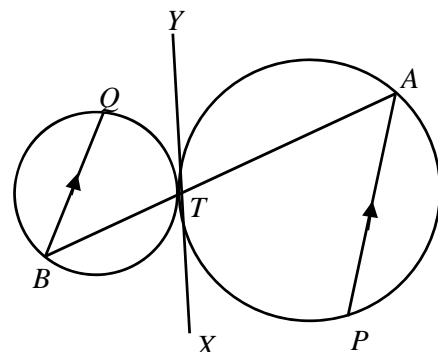
(d) Draw the common tangent XY to the circles at T

$\angle TAP = \angle XTP$ and $\angle QBT = \angle YTQ$ (angle between a chord and a tangent is equal to the angle subtending the alternate segment)

But $\angle TAP = \angle QBT$ (alternate angles on parallel lines)

$$\therefore \angle XTP = \angle YTQ$$

$\therefore Q, T, P$ are collinear (vertically opposite angles)



Question 14

(a) (i) For $k > 0, (k+1)^2 > (k(k+1))$

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\therefore \frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

(ii) Let $n = 2$, LHS = $\frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4}$, RHS = $2 - \frac{1}{2} = \frac{3}{2}$

\therefore true, since $\frac{5}{4} < \frac{3}{2}$.

Assume $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for some integer n

RTP $\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}$

$$\text{LHS} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

$$< 2 - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}, \text{ from part (i)}$$

$$= 2 - \frac{1}{n+1} = \text{RHS}, \therefore \text{true.}$$

\therefore By the principle of Induction, it's true for all $n \geq 2$

(b) (i) $\binom{4n}{2n}$

(ii) $(1 + (x^2 + 2x))^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} 1^k (x^2 + 2x)^{2n-k}$
 $= \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} (x+2)^{2n-k}$, as $x^2 + 2x = x(x+2)$.

(iii) $1 + x^2 + 2x = (1+x)^2, \therefore (1 + x^2 + 2x)^{2n} = (1+x)^{4n}$

$\therefore \binom{4n}{2n}$ is the coefficient of x^{2n} in $(1+x)^{4n}$.

Given $x^{2n-k} (x+2)^{2n-k} = \binom{2n-k}{0} 2^{2n-k} x^{2n-k} +$

$$\binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} + \dots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k},$$

$$\sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} (x+2)^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k} \times$$

$$\left\{ \binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} + \dots \right.$$

$$\left. + \binom{2n-k}{2n-k} 2^0 x^{4n-2k} \right\}$$

$$\text{In } \binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} +$$

$+\dots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k}$, the coefficient of x^{2n} is

$$\binom{2n}{0} 2^{2n}, \text{ when } k=0,$$

$$+ \binom{2n-1}{1} 2^{2n-2}, \text{ when } k=1,$$

$$+ \dots + \binom{2n-n}{n} 2^0, \text{ when } k=n.$$

\therefore The coefficient of x^{2n} is

$$\binom{2n}{0} \binom{2n}{0} 2^{2n} + \binom{2n}{1} \binom{2n-1}{1} 2^{2n-2} + \dots$$

$$+ \binom{2n}{n} \binom{2n-n}{n} 2^0.$$

$$\therefore \binom{4n}{2n} = \sum_{k=0}^n \binom{2n}{k} \binom{2n-k}{k} 2^{2n-2k}.$$

(c) (i) Let $f(t) = e^t - \frac{1}{t}$

$$f'(t) = e^t + \frac{1}{t^2}$$

$$t_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{e^{0.5} - 2}{e^{0.5} + 4} = 0.562 \approx 0.56$$

(ii) When two curves $g(x)$ and $h(x)$ have a common tangent at their point of intersection, $g(x) - h(x)$ yields a double root at the point of contact.

Let $f(x) = e^x - \ln(x) + C$

$$f'(x) = e^x - \frac{1}{x}.$$

From (i), when $x \approx 0.56, f'(x) \approx 0$

\therefore We want $f(0.56) \approx 0$, too, $e^{0.56} - \ln 0.56 + C = 0$

$$\therefore C = \ln 0.56 - e^{0.56} \approx -2.33$$

$$\therefore f(x) = e^x - \ln x - 2.33$$

Let $2.33 = \ln C, \therefore C = e^{2.33} \approx 10.3,$

$$\therefore f(x) = e^x - \ln(10.3x)$$

\therefore The curves e^x and $\ln(10.3x)$ have a common tangent at their point of intersection $x \approx 0.56$.

\therefore The curves $e^{\frac{x}{10.3}}$ and $\ln x$ have a common tangent at their point of intersection $x \approx 5.77, \text{ i.e. } r \approx \frac{1}{10.3}$

$$= 0.097$$