Multiple Choice

1) (D) 80, angle at centre is twice angle at circumference

2) (A)

3) (C), $\left(2x-\frac{5}{x^3}\right)^{12}=\frac{(2x^4-5)^{12}}{x^{36}},\therefore\ {}^{12}C_9 2^9(-5)^3$

4) (D), $\tan\theta=\frac{-1-3}{1+(-1)(3)}=\frac{-4}{-2}=2$

5) (B), $\prod\alpha=-42,\sum\alpha\beta=-41$

6) (B)

7) (A) $6=\frac{2\pi}{n},\therefore n=\frac{\pi}{3},\therefore x=5\sin\frac{\pi t}{3},\therefore v=\frac{5\pi}{3}\cos\frac{\pi t}{3}$

8) (D), ${}^{15}C_6 5!=\frac{15!5!}{6!9!}=\frac{15!}{6\times 9!}$

9) (C), $x^4-8x^3-7x^2+3=x(x+1)Q(x)+ax+3$

Put $x=-1, 1+8-7+3=-a+3,\therefore a=-2$

10) (C), it is the perpendicular bisector of the 2 points,

$m_1=\frac{b+a-(b-a)}{a-b-(a+b)}=-\frac{a}{b},\therefore m_2=\frac{b}{a}$

Midpoint $=\left(\frac{a+b+a-b}{2},\frac{b-a+b+a}{2}\right)=(a,b)$

$y-b=\frac{b}{a}(x-a),\therefore ay=bx$

Question 11

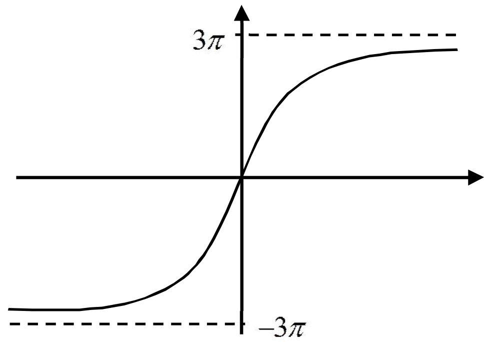
(a) Let $x+\frac{2}{x}=u, u^2-6u+9=(u-3)^2=0,$

$\therefore u=3,\therefore x+\frac{2}{x}=3,\therefore x^2-3x+2=(x-1)(x-2)=0$

$\therefore x=1,2$

(b) $\Pr=0.9^{30}+{}^{30}C_1 0.1\times 0.9^{29}+{}^{30}C_2 0.1^2\times 0.9^{28}$

(c)



(d) $\int_2^5\frac{x}{\sqrt{x-1}}dx$

Let $x=u^2+1, dx=2udu.$

When $x=2, u=1$; when $x=5, u=2$

$\int_1^2\frac{u^2+1}{u}2udu=2\int_1^2(u^2+1)du=2\left[\frac{u^3}{3}+u\right]_1^2$

$=2\left(\frac{8}{3}-\frac{1}{3}\right)+2(2-1)=\frac{14}{3}+2=\frac{20}{3}$

(e) $\frac{x^2+5}{x}>6,\therefore x^3+5x>6x^2,\therefore x^3-6x^2+5x>0$

$x(x^2-6x+5)=x(x-1)(x-5)>0$

$\therefore 0<x<1, x>5$

(f) $y=\frac{e^x\ln x}{x},$

$y'=\frac{e^x\left(\ln x+\frac{1}{x}\right)x-e^x\ln x}{x^2}=\frac{(x-1)e^x\ln x+e^x}{x^2}$

Question 12

(a) (i) 4 m

(ii) $v = 6\cos 3t, a = -18\sin 3t$.

$\therefore$ When it's first at rest, $\cos 3t = 0, \sin 3t = 1$,

$\therefore a = -18 \text{ m/s}^2$

(b) $V = \pi \int_0^{\frac{\pi}{8}} \cos^2 4x \, dx = \frac{\pi}{2} \int_0^{\frac{\pi}{8}} (1 + \cos 8x) \, dx$

$= \frac{\pi}{2}\left[x + \frac{\sin 8x}{8}\right]_0^{\frac{\pi}{8}} = \frac{\pi}{2}\left(\frac{\pi}{8}\right) = \frac{\pi^2}{16}$

(c) $\ddot{x} = 2 - e^{-\frac{x}{2}} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = 2x + 2e^{-\frac{x}{2}} + C$

$v = 4, x = 0, \therefore 8 = 2 + C, \therefore C = 6$

$\therefore v^2 = 2\left(2x + 2e^{-\frac{x}{2}} + 6\right) = 4\left(x + e^{-\frac{x}{2}} + 3\right)$

(d) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$

Let $x = -1, 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n}$

(e) The tangent to the curve at $x = x_1$ meets the x-axis at a point further to α than x_1 is.

(f) $T = 23 - Be^{-0.03t}$

When $t = 0, T = 2, \therefore 2 = 23 - B, \therefore B = 21$

$\therefore T = 23 - 21e^{-0.03t}$

When $T = 10, 10 = 23 - 21e^{-0.03t}$

$e^{-0.03t} = \frac{13}{21}$

$-0.03t = \ln\frac{13}{21}$

$\therefore t = \frac{\ln\frac{13}{21}}{-0.03} \simeq 16$ minutes.

Question 13

(a) Let $n = 1, 2 + 1 = 3$ is divisible by 3, $\therefore$ true.

Assume $2^n + (-1)^{n+1} = 3M$, where M is an integer

$\therefore 2^n = 3M - (-1)^{n+1} = 3M + (-1)^{n+2}$.

Required to prove that $2^{n+1} + (-1)^{n+2}$ is divisible by 3.

$2^{n+1} + (-1)^{n+2} = 2\left(3M + (-1)^{n+2}\right) + (-1)^{n+2}$

$= 6M + 3(-1)^{n+2}$

$= 3\left(2M + (-1)^{n+2}\right)$, which is divisible by 3.

$\therefore 2^n + (-1)^{n+1}$ is divisible by 3 for all $n \geq 1$.

(b) (i) $L^2 = 40^2 + x^2$

$2L\frac{dL}{dx} = 2x$

$\frac{dL}{dx} = \frac{x}{L} = \cos\theta$

(ii) $\frac{dL}{dt} = \frac{dx}{dt}\frac{dL}{dx} = 3\cos\theta$

(c) (i) $P(2at, at^2), S(0, a)$

$$Q\begin{cases} x = \dfrac{1 \times 2at + t^2 \times 0}{1+t^2} = \dfrac{2at}{1+t^2} \\ y = \dfrac{1 \times at^2 + t^2 \times a}{1+t^2} = \dfrac{2at^2}{1+t^2} \end{cases}$$

(ii) $m_{OQ} = \dfrac{\frac{2at^2}{1+t^2}}{\frac{2at}{1+t^2}} = t$

(iii) $m_P = \frac{dy/dt}{dx/dt} = \frac{2at}{2a} = t$

Since the gradient of the tangent at P is also t, OQ // the tangent at P.

But from the optical property of the parabola, $SP = ST$, where T is the point the tangent at P meets the y-axis, $\therefore \Delta SPT \,|||\, \Delta SQO, \therefore SQ = SO = a$.

$\therefore$ The locus of Q is a circle of centre S, radius a.

(d) (i) In a cyclic quad, the interior angle is equal to the opposite exterior angle

(ii) $\angle OPA = \angle OAP$ ($OP = OA =$ radius, $\therefore \Delta OPA$ is isosceles)

Extend OP to Q, $\angle OPA = \angle CPQ$ (vertically opposite)

$\therefore \angle CPQ = \angle CQP$ (since both $= \angle OAP$)

$\therefore OP$ is a tangent to the circle CPQ (angles in alternate segments are equal)

Question 14

(a) (i) $x = Vt\cos\theta, \therefore t = \dfrac{x}{V\cos\theta}$

$$\therefore y = -\frac{1}{2}g\frac{x^2}{V^2\cos^2\theta} + V\frac{x}{V\cos\theta}\sin\theta$$
$$= x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$
$$= x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

(ii) P is the point on the line $y = -x$

$$-x = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$
$$-1 = \tan\theta - \frac{gx}{2V^2}\sec^2\theta$$
$$\frac{gx}{2V^2}\sec^2\theta = 1 + \tan\theta$$
$$\therefore x = \frac{2V^2}{g}\frac{1+\tan\theta}{\sec^2\theta} = \frac{2V^2}{g}\left(\cos^2\theta + \sin\theta\cos\theta\right)$$
$$= \frac{2V^2}{g}\cos\theta\left(\cos\theta + \sin\theta\right)$$
$$\therefore D = \sqrt{2}x = \frac{2\sqrt{2}V^2}{g}\cos\theta\left(\cos\theta + \sin\theta\right)$$

(iii) $D = \dfrac{2\sqrt{2}V^2}{g}\left(\cos^2\theta + \sin\theta\cos\theta\right)$

$$= \frac{2\sqrt{2}V^2}{g}\left(\frac{1+\cos 2\theta}{2} + \frac{\sin 2\theta}{2}\right)$$
$$\frac{dD}{d\theta} = \frac{2\sqrt{2}V^2}{g}\left(-\sin 2\theta + \cos 2\theta\right)$$

(iv) $\dfrac{dD}{d\theta} = 0$ when $\sin 2\theta = \cos 2\theta, \therefore \tan 2\theta = 1$

$$\therefore 2\theta = \frac{\pi}{4}, \therefore \theta = \frac{\pi}{8}.$$
$$\frac{d^2D}{d\theta^2} = \frac{4\sqrt{2}V^2}{g}\left(-\cos 2\theta - \sin 2\theta\right)$$
$$= \frac{4\sqrt{2}V^2}{g}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -\frac{8V^2}{g} < 0 \text{ when } \theta = \frac{\pi}{8}$$

$\therefore D$ is maximum when $\theta = \dfrac{\pi}{8}$.

(b) (i) Player A takes the first turn.

If the arrow stops in section P, he wins, $\Pr = p$

If the arrow stops in section R, and player B's arrow stops in section Q, player A wins, $\Pr = rq$

$$\therefore \Pr = p + rq$$
$$= p + r(1-p-r), \text{ since } p+q+r=1$$
$$= p + r - rp - r^2$$
$$= (p+r) - r(p+r)$$
$$= (1-r)(p+r)$$

(ii) The probabilities that player A wins in the first, second, third, and fourth turns are $p, rq, r^2p, r^3q, \ldots$

$\therefore$ The probability that he eventually wins the game is

$$p + rq + r^2p + r^3q + r^4p + r^5q + \ldots$$
$$= (1-r)(p+r) + r^2(1-r)(p+r) + r^4(1-r)(p+r) + \ldots,$$

since $p + rq = (1-r)(p+r)$

$$= (1-r)(p+r)\left(1 + r^2 + r^4 + \ldots\right)$$
$$= (1-r)(p+r)\frac{1}{1-r^2}$$
$$= \frac{p+r}{1+r}$$