Multiple Choice

1) (A) $(-3)^3+18=-9$

2) (C) $\frac{dN}{dt}=k\times 80e^{kt}=80(N-100)$

3) (B), $x(x+3)=4\times 10, x^2+3x-40=(x+8)(x-5), \therefore =5$

4) (C), ${}^{12}C_8\times {}^{4}C_1$

5) (A), $x=-1,-2, y=0$

6) (D) $-1\le 2x\le 1, \therefore -\frac{1}{2}\le x\le \frac{1}{2}$

7) (B) $\sin^{-1}\frac{k}{2}=\frac{\pi}{3}, \therefore \frac{k}{2}=\frac{\sqrt{3}}{2}, \therefore k=\sqrt{3}$

8) (B), $\lim_{x\to 3}\frac{\sin(x-3)}{x-3}=1, \therefore \lim_{x\to 3}\frac{\sin(x-3)}{(x-3)(x+2)}=\frac{1}{5}$

9) (D), $a=6, T=\frac{2\pi}{n}\alpha\frac{1}{n}, \therefore n=8$

10) (C), $\cos\left(2t-\frac{\pi}{3}\right)=0, \therefore 2t-\frac{\pi}{3}=\frac{\pi}{2}+\pi$

$2t=\frac{5\pi}{6}+\pi=\frac{11\pi}{6}, \therefore t=\frac{11\pi}{12}$

Question 11

(a) $\int \sin^2 x\,dx=\int\frac{1-\cos 2x}{2}dx=\frac{x}{2}-\frac{\sin 2x}{4}+C$

(b) $\tan\alpha=\left|\frac{2-(-3)}{1+2\times(-3)}\right|=1, \therefore \alpha=45°$

(c) $\frac{4}{x+3}\ge 1$

$4(x+3)\ge (x+3)^2$

$(x+3)(4-x-3)\ge 0$

$(x+3)(1-x)\ge 0$

$\therefore -3<x\le 1$

(d) $5\cos x-12\sin x=13\cos\left(x+\tan^{-1}\frac{12}{5}\right)$

$\simeq 13\cos(x+1.176)$

(e) $u=2x-1, du=2dx$ and $x=\frac{u+1}{2}$

$\int_1^2\frac{x}{(2x-1)^2}du=\frac{1}{4}\int_1^3\frac{(u+1)}{u^2}du$

$=\frac{1}{4}\int_1^3\left(\frac{1}{u}+\frac{1}{u^2}\right)du$

$=\frac{1}{4}\left[\ln u-\frac{1}{u}\right]_1^3$

$=\frac{1}{4}\left(1-\frac{1}{3}+\ln 3\right)$

$=\frac{1}{4}\left(\frac{2}{3}+\ln 3\right)$

(f) (i) $P(3)=0, \therefore 27-9k+15+12=0, \therefore k=6$

(ii) $x^3-6x^2+5x+12=(x-3)(x^2-3x-4)$

$=(x-3)(x-4)(x+1)$

$\therefore x=3,4,-1.$

Question 12

(a)(i) $\angle ACB = 90° - 30° = 60°$ (semi-circle angle $= 90°$)

(ii) $\angle ADX = 30°$ (angle between the tangent and a chord is equal angle in the alternate segment)

(iii) $\angle CDB = 70°$ (exterior angle in a triangle is equal to the sum of two opposite interior angles)

$\angle CAB = 70°$ (angles subtending the same arc are equal)

(b) (i) Sub $(0,a)$ to the equation of PQ,

$-2a - 2apq = 0$

If this statement is true then $pq = -1$.

$\therefore$ If PQ is a focal chord then $pq = -1$.

(ii) $8a = 2ap, \therefore p = 4$

$\therefore q = -\frac{1}{4}, \therefore Q(2aq, aq^2)$ becomes $\left(-\frac{a}{2}, \frac{a}{16}\right)$.

(c) (i) $\tan 15° = \frac{h}{OA}, \therefore OA = h\cot 15°$.

(ii) Similarly, $OB = h\cot 13°$

$AB^2 = OB^2 - OA^2$

$2000^2 = h^2\left(\cot^2 13° - \cot^2 15°\right)$

$\therefore h = \frac{2000}{\sqrt{\cot^2 13° - \cot^2 15°}} = 909.7$ m

(d)(i) By the Cosine rule,

$160^2 = r^2 + r^2 - 2r \times r \times \cos\theta = 2r^2(1 - \cos\theta)$

(ii) Arc length $200 = r\theta, \therefore r = \frac{200}{\theta}$

$\therefore 160^2 = 2 \times \frac{200^2}{\theta^2}(1 - \cos\theta)$

$8\theta^2 = 25(1 - \cos\theta)$

$\therefore 8\theta^2 + 25\cos\theta - 25 = 0$

(iii) $f(\theta) = 8\theta^2 + 25\cos\theta - 25$

$f'(\theta) = 16\theta - 25\sin\theta$

$\therefore \theta_1 = \pi - \frac{8\pi^2 + 25\cos\pi - 25}{16\pi - 25\sin\pi}$

$= \pi - \frac{8\pi^2 - 50}{16\pi} = 2.57$ (to 2 dp)

Question 13

(a) (i) $x = 3$ or 7 m

(ii) $\sqrt{11}$ m/s

(iii) $a = \text{amplitude} = 2$

$c = \text{centre} = 5$

When $x = 5, v^2 = 11, \therefore 11 = 4n^2, \therefore n = \frac{\sqrt{11}}{2}$

(b) (i) $\left(2x + \frac{1}{3x}\right)^{18} = \frac{(6x^2+1)^{18}}{(3x)^{18}}$.

Coefficient of x^{14} = coefficient of $(x^2)^{16}$ in $(6x^2+1)^{18}$

$\therefore$ It's $= \frac{{}^{18}C_{16}6^{16}}{3^{18}} = \frac{{}^{18}C_{16}2^{16}}{3^2}$.

(ii) The term independent of x = coefficient of $(x^2)^9$ in $(6x^2+1)^{18}$.

$\therefore$ It's $= \frac{{}^{18}C_9 6^9}{3^{18}} = \frac{{}^{18}C_9 2^9}{3^9}$

(c) Let $n = 1$, LHS $= \frac{1}{2}$, RHS $= 1 - \frac{1}{2} = \frac{1}{2}$. $\therefore$ It's true for $n = 1$.

Assume $\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

RTP $\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$

LHS $= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$

$= 1 - \frac{(n+2)-(n+1)}{(n+2)!}$

$= 1 - \frac{1}{(n+2)!} =$ RHS.

$\therefore$ It's true for all integers $n \geq 1$.

(d)(i) $f'(x) = \frac{-1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-(-x)^2}} \times \frac{d}{dx}(-x)$

$= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0, \therefore f(x)$ is a constant.

Let $x = 0$, $f(0) = \cos^{-1}0 + \cos^{-1}0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

$\therefore \cos^{-1}x + \cos^{-1}(-x) = \pi$

$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}x$

Question 14

(a) (i) When $y=0, Vt\sin\theta-\frac{1}{2}gt^2=0, \therefore t=\frac{2V\sin\theta}{g}$

Sub to $x, x=V\frac{2V\sin\theta}{g}\cos\theta=\frac{V^2\sin 2\theta}{g}$.

(ii) $\dot{x}=V\cos\theta=V\cos\frac{\pi}{3}=\frac{V}{2}$

$\dot{y}=V\sin\theta-gt=V\sin\frac{\pi}{3}-gt=\frac{\sqrt{3}V}{2}-g\frac{2V}{\sqrt{3}g}=\frac{-V}{2\sqrt{3}}$

$\frac{\dot{y}}{\dot{x}}=\frac{\frac{-V}{2\sqrt{3}}}{\frac{V}{2}}=-\frac{1}{\sqrt{3}}, \therefore \text{Angle}=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

(iii) The negative sign above indicates that the particle is travelling downwards.

(b) (i) $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)=x-1$

$\frac{1}{2}\dot{x}^2=\frac{x^2}{2}-x+C$

When $x=0, \dot{x}=1, \therefore C=\frac{1}{2}$.

$\therefore \frac{1}{2}\dot{x}^2=\frac{x^2}{2}-x+\frac{1}{2}$

$\therefore \dot{x}^2=x^2-2x+1=(x-1)^2$

$\therefore \dot{x}=\pm(x-1)$.

When $x=0, \dot{x}=1, \therefore$ take $\dot{x}=1-x$.

(ii) $\dot{x}=\frac{dx}{dt}=1-x$

$\frac{dt}{dx}=\frac{1}{1-x}$

$t=-\ln(1-x)+C$

$t=0, x=0, \therefore C=0$

$\therefore t=-\ln(1-x)$

$\therefore 1-x=e^{-t}$

$\therefore x=1-e^{-t}$

(iii) When $t\to\infty, x\to 1$.

The limiting position is $x=1$.

(c) (i) Pr(win 5 in 7 games) = Pr(win 4 in 6 games and win the last game) $=\binom{6}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2\times\frac{1}{2}$

$=\binom{6}{4}\left(\frac{1}{2}\right)^7$

(ii) Pr(win 5 in at most 7 games) = Pr(win 5 in 5 games)+ Pr(win 5 in 6 games)+ Pr(win 5 in 7 games)

$=\binom{4}{4}\left(\frac{1}{2}\right)^5+\binom{5}{4}\left(\frac{1}{2}\right)^6+\binom{6}{4}\left(\frac{1}{2}\right)^7$

(iii) Pr(one player wins $(n+1)$ games)

$=\binom{n}{n}\left(\frac{1}{2}\right)^{n+1}+\binom{n+1}{n}\left(\frac{1}{2}\right)^{n+2}+\binom{n+2}{n}\left(\frac{1}{2}\right)^{n+3}+\dots$

$+\binom{2n}{n}\left(\frac{1}{2}\right)^{2n+1}$, as after $2n+1$ games there must be a winner,

$=\frac{1}{2}$, as each player has an equal chance.

Multiplying both sides by 2^{2n+1},

$\binom{n}{n}2^n+\binom{n+1}{n}2^{n-1}+\binom{n+2}{n}2^{n-2}+\dots+\binom{2n}{n}=2^{2n}$.