

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–13

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

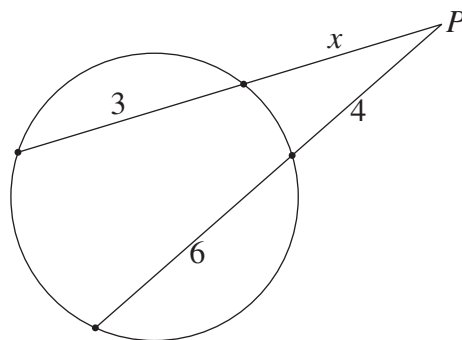
10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

- 1 What is the remainder when $x^3 - 6x$ is divided by $x + 3$?
- (A) -9
(B) 9
(C) $x^2 - 2x$
(D) $x^2 - 3x + 3$
- 2 Given that $N = 100 + 80e^{kt}$, which expression is equal to $\frac{dN}{dt}$?
- (A) $k(100 - N)$
(B) $k(180 - N)$
(C) $k(N - 100)$
(D) $k(N - 180)$
- 3 Two secants from the point P intersect a circle as shown in the diagram.



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What is the value of x ?

- (A) 2
(B) 5
(C) 7
(D) 8

4 A rowing team consists of 8 rowers and a coxswain.

The rowers are selected from 12 students in Year 10.

The coxswain is selected from 4 students in Year 9.

In how many ways could the team be selected?

(A) ${}^{12}C_8 + {}^4C_1$

(B) ${}^{12}P_8 + {}^4P_1$

(C) ${}^{12}C_8 \times {}^4C_1$

(D) ${}^{12}P_8 \times {}^4P_1$

5 What are the asymptotes of $y = \frac{3x}{(x+1)(x+2)}$?

(A) $y = 0, x = -1, x = -2$

(B) $y = 0, x = 1, x = 2$

(C) $y = 3, x = -1, x = -2$

(D) $y = 3, x = 1, x = 2$

6 What is the domain of the function $f(x) = \sin^{-1}(2x)$?

(A) $-\pi \leq x \leq \pi$

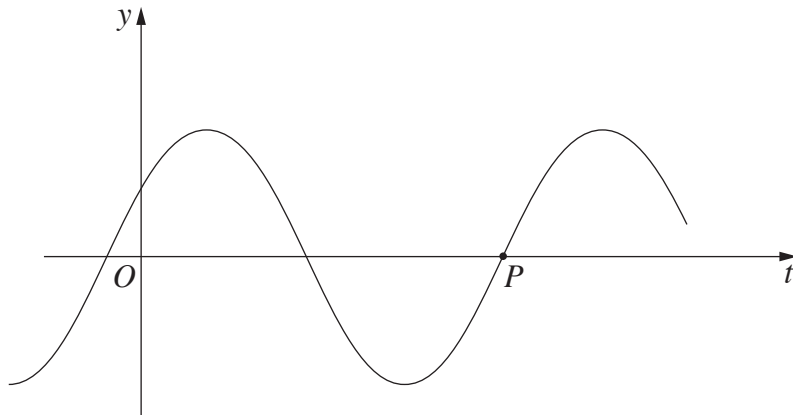
(B) $-2 \leq x \leq 2$

(C) $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

(D) $-\frac{1}{2} \leq x \leq \frac{1}{2}$

- 7 What is the value of k such that $\int_0^k \frac{1}{\sqrt{4-x^2}} dx = \frac{\pi}{3}$?
- (A) 1
(B) $\sqrt{3}$
(C) 2
(D) $2\sqrt{3}$
- 8 What is the value of $\lim_{x \rightarrow 3} \frac{\sin(x-3)}{(x-3)(x+2)}$?
- (A) 0
(B) $\frac{1}{5}$
(C) 5
(D) Undefined
- 9 Two particles oscillate horizontally. The displacement of the first is given by $x = 3 \sin 4t$ and the displacement of the second is given by $x = a \sin nt$. In one oscillation, the second particle covers twice the distance of the first particle, but in half the time.
- What are the values of a and n ?
- (A) $a = 1.5, n = 2$
(B) $a = 1.5, n = 8$
(C) $a = 6, n = 2$
(D) $a = 6, n = 8$

- 10 The graph of the function $y = \cos\left(2t - \frac{\pi}{3}\right)$ is shown below.



What are the coordinates of the point P ?

- (A) $\left(\frac{5\pi}{12}, 0\right)$
- (B) $\left(\frac{2\pi}{3}, 0\right)$
- (C) $\left(\frac{11\pi}{12}, 0\right)$
- (D) $\left(\frac{7\pi}{6}, 0\right)$

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \sin^2 x \, dx$. **2**

(b) Calculate the size of the acute angle between the lines $y = 2x + 5$ and $y = 4 - 3x$. **2**

(c) Solve the inequality $\frac{4}{x+3} \geq 1$. **3**

(d) Express $5 \cos x - 12 \sin x$ in the form $A \cos(x + \alpha)$, where $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

(e) Use the substitution $u = 2x - 1$ to evaluate $\int_1^2 \frac{x}{(2x-1)^2} \, dx$. **3**

(f) Consider the polynomials $P(x) = x^3 - kx^2 + 5x + 12$ and $A(x) = x - 3$.

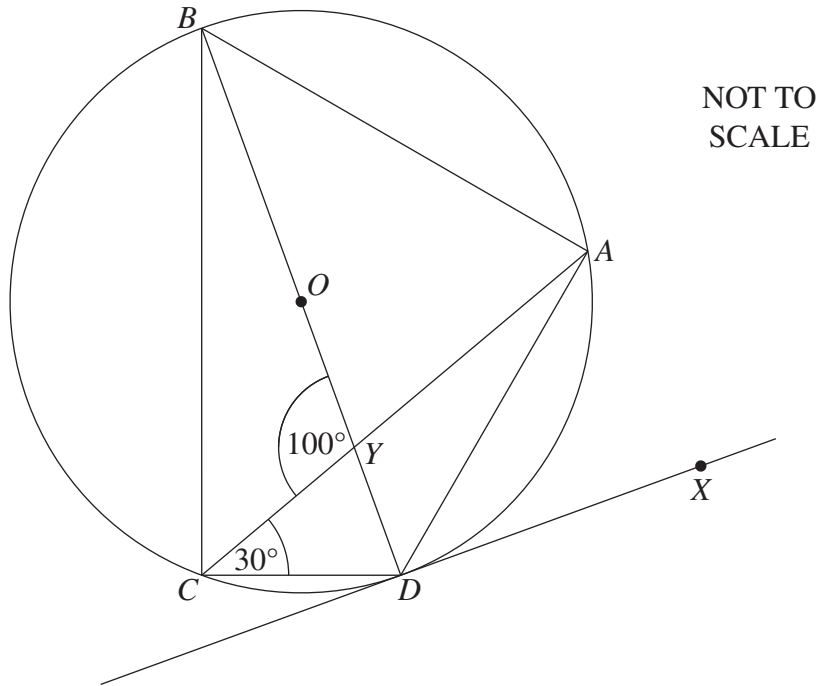
(i) Given that $P(x)$ is divisible by $A(x)$, show that $k = 6$. **1**

(ii) Find all the zeros of $P(x)$ when $k = 6$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

- (a) In the diagram, the points A , B , C and D are on the circumference of a circle, whose centre O lies on BD . The chord AC intersects the diameter BD at Y . The tangent at D passes through the point X .

It is given that $\angle CYB = 100^\circ$ and $\angle DCY = 30^\circ$.



Copy or trace the diagram into your writing booklet.

- | | |
|--|----------|
| (i) What is the size of $\angle ACB$? | 1 |
| (ii) What is the size of $\angle ADX$? | 1 |
| (iii) Find, giving reasons, the size of $\angle CAB$. | 2 |

Question 12 continues on page 8

Question 12 (continued)

- (b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

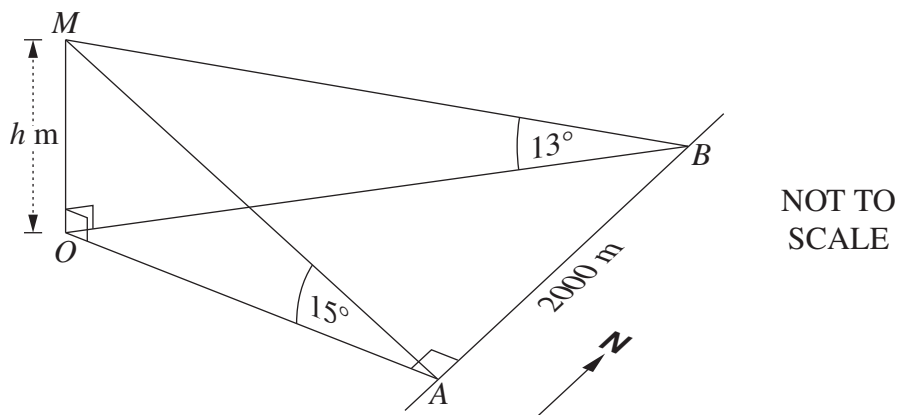
The equation of the chord PQ is given by $(p + q)x - 2y - 2apq = 0$. (Do NOT prove this.)

- (i) Show that if PQ is a focal chord then $pq = -1$. 1
- (ii) If PQ is a focal chord and P has coordinates $(8a, 16a)$, what are the coordinates of Q in terms of a ? 2

- (c) A person walks 2000 metres due north along a road from point A to point B . The point A is due east of a mountain OM , where M is the top of the mountain. The point O is directly below point M and is on the same horizontal plane as the road. The height of the mountain above point O is h metres.

From point A , the angle of elevation to the top of the mountain is 15° .

From point B , the angle of elevation to the top of the mountain is 13° .

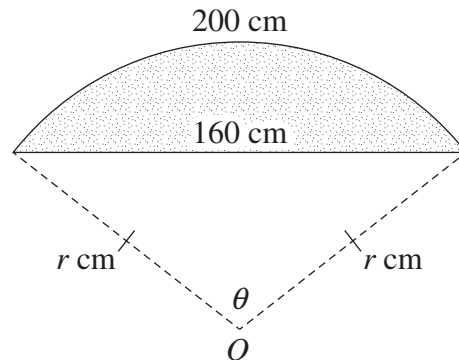


- (i) Show that $OA = h \cot 15^\circ$. 1
- (ii) Hence, find the value of h . 2

Question 12 continues on page 9

Question 12 (continued)

- (d) A kitchen bench is in the shape of a segment of a circle. The segment is bounded by an arc of length 200 cm and a chord of length 160 cm. The radius of the circle is r cm and the chord subtends an angle θ at the centre O of the circle.



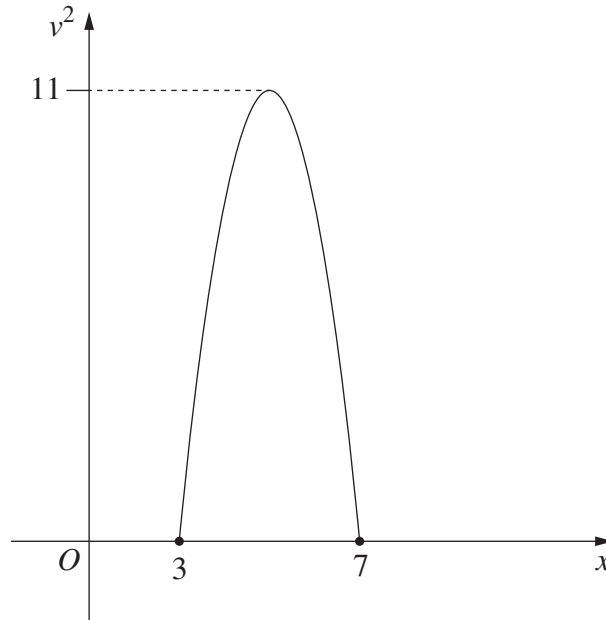
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- (i) Show that $160^2 = 2r^2(1 - \cos \theta)$. 1
- (ii) Hence, or otherwise, show that $8\theta^2 + 25 \cos \theta - 25 = 0$. 2
- (iii) Taking $\theta_1 = \pi$ as a first approximation to the value of θ , use one application of Newton's method to find a second approximation to the value of θ . Give your answer correct to two decimal places. 2

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

- (a) A particle is moving along the x -axis in simple harmonic motion. The displacement of the particle is x metres and its velocity is $v \text{ m s}^{-1}$. The parabola below shows v^2 as a function of x .



- (i) For what value(s) of x is the particle at rest? **1**
- (ii) What is the maximum speed of the particle? **1**
- (iii) The velocity v of the particle is given by the equation **3**

$$v^2 = n^2(a^2 - (x - c)^2)$$

where a , c and n are positive constants.

What are the values of a , c and n ?

Question 13 continues on page 11

Question 13 (continued)

(b) Consider the binomial expansion

$$\left(2x + \frac{1}{3x}\right)^{18} = a_0x^{18} + a_1x^{16} + a_2x^{14} + \dots$$

where a_0, a_1, a_2, \dots are constants.

(i) Find an expression for a_2 . **2**

(ii) Find an expression for the term independent of x . **2**

(c) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

(d) Let $f(x) = \cos^{-1}(x) + \cos^{-1}(-x)$, where $-1 \leq x \leq 1$.

(i) By considering the derivative of $f(x)$, prove that $f(x)$ is constant. **2**

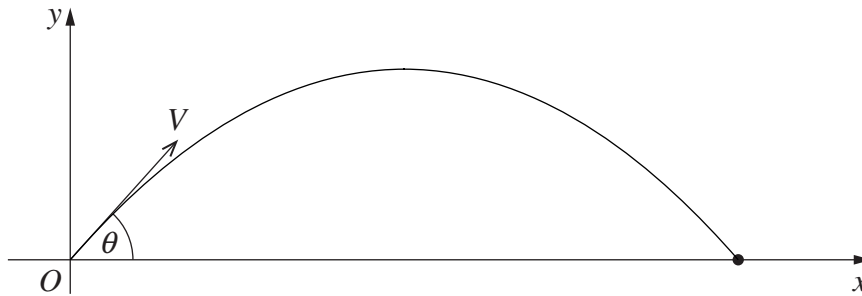
(ii) Hence deduce that $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

- (a) A projectile is fired from the origin O with initial velocity $V \text{ m s}^{-1}$ at an angle θ to the horizontal. The equations of motion are given by

$$x = Vt \cos \theta, \quad y = Vt \sin \theta - \frac{1}{2}gt^2. \quad (\text{Do NOT prove this.})$$



- (i) Show that the horizontal range of the projectile is $\frac{V^2 \sin 2\theta}{g}$. **2**

A particular projectile is fired so that $\theta = \frac{\pi}{3}$.

- (ii) Find the angle that this projectile makes with the horizontal when **2**
 $t = \frac{2V}{\sqrt{3}g}$.

- (iii) State whether this projectile is travelling upwards or downwards when **1**
 $t = \frac{2V}{\sqrt{3}g}$. Justify your answer.

Question 14 continues on page 13

Question 14 (continued)

- (b) A particle is moving horizontally. Initially the particle is at the origin O moving with velocity 1 m s^{-1} .

The acceleration of the particle is given by $\ddot{x} = x - 1$, where x is its displacement at time t .

- (i) Show that the velocity of the particle is given by $\dot{x} = 1 - x$. **3**
- (ii) Find an expression for x as a function of t . **2**
- (iii) Find the limiting position of the particle. **1**
- (c) Two players A and B play a series of games against each other to get a prize. In any game, either of the players is equally likely to win.

To begin with, the first player who wins a total of 5 games gets the prize.

- (i) Explain why the probability of player A getting the prize in exactly 7 games is $\binom{6}{4} \left(\frac{1}{2}\right)^7$. **1**
- (ii) Write an expression for the probability of player A getting the prize in at most 7 games. **1**
- (iii) Suppose now that the prize is given to the first player to win a total of $(n + 1)$ games, where n is a positive integer. **2**

By considering the probability that A gets the prize, prove that

$$\binom{n}{n}2^n + \binom{n+1}{n}2^{n-1} + \binom{n+2}{n}2^{n-2} + \dots + \binom{2n}{n} = 2^{2n}.$$

End of paper