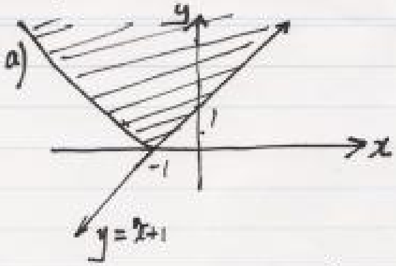
HSC - <u>Mathematics Extension</u> 1 <u>exam</u> <u>SOLUTIONS</u>
2004 Jan Hansen

① a)

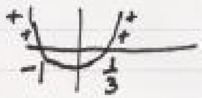


b) $\frac{4}{(x+1)} \times (x+1)^2 < 3(x+1)^2$

$4x+4 < 3(x^2+2x+1)$

$3x^2+2x-1 > 0$

$(3x-1)(x+1) > 0$



$\therefore\ x<-1,\ x>\frac{1}{3}$

c) $x = \frac{5\cdot 9 - 2.3}{5-2}$

$y = \frac{5.2 - 2.-1}{5-2}$

$\therefore P(13, 4)$

d) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\Big|_0^1$

$= \sin^{-1}\frac{1}{2} - 0$

$= \frac{\pi}{6}$

e) $u = x-3$, $du = dx$ $\quad x=4, u=1;\ x=3, u=0$

$I = \int_0^1 (u+3)\,u^{1/2}\,du$

$= \int_0^1 (u^{3/2} + 3u^{1/2})\,du$

$= \frac{2}{5}u^{5/2} + 2u^{3/2}\Big|_0^1$

$= 2/5 + 2 = \frac{12}{5}$.

② (a) $\lim_{x\to 0} \frac{\sin\frac{x}{5}}{2x} = \frac{1}{10}\lim_{x\to 0}\frac{\sin\frac{x}{5}}{x/5}$

$= \frac{1}{10}$

b) $\frac{d}{dx}\left(\cos^{-1}(3x^2)\right)$

$= \frac{-1}{\sqrt{1-9x^4}} \times 6x$

$= \frac{-6x}{\sqrt{1-9x^4}}$

c) $AT^2 = TC.TB$ circle result.

$\therefore x = AT^2/TB$

$x = 12^2/(x+7)$

$x^2+7x-144 = 0$

$x = (-7 \pm 25)/2$

$\underline{x = 9}$ (as $x>0$)

d) i) $A\cos(x-\alpha) = A\cos\alpha\cos x + A\sin\alpha\sin x$

$\therefore\ 8 = A\cos\alpha,\ 6 = A\sin\alpha$

$A^2 = 64+36$

$\therefore A = 10$.

$\tan\alpha = 3/4$

$\alpha = \tan^{-1}\frac{3}{4}$

So $8\cos x + 6\sin x = 10\cos\left(x - \tan^{-1}\frac{3}{4}\right)$

ii) $10\cos(x-\alpha) = 5$

$\cos(x-\alpha) = \frac{1}{2}$

$x-\alpha = 2\pi n \pm \frac{\pi}{3}$ $\quad n\in\mathbb{Z}$.

$x = \tan^{-1}\frac{3}{4} + \frac{5\pi}{3},\ \tan^{-1}\frac{3}{4}+\frac{\pi}{3}$

$x = 5.879, 1.691$

e) i) ${}^{16}C_4 = 1820$

ii) ${}^{9}C_4 / {}^{16}C_4 = \frac{9}{130}$

③ a) $\int \cos^2 4x\,dx = \frac{1}{2}\int(1+\cos 8x)\,dx$

$= \frac{1}{2}\left(x + \frac{\sin 8x}{8}\right) + c$

b) i) $P(-1) = -11$, $P(3) = 1$

$\therefore b = -11$

and $4a+b = 1$

$a = 3$

ii) $R(x) = 3(x+1) + -11$

$= 3x-8$

c) i) $x^2+h^2 = 16$

$x = \sqrt{16-h^2}$

ii) when $h=1$, $\frac{dh}{dt} = -0.3$ m/hr

$\frac{dx}{dt} = \frac{-hh'}{\sqrt{16-h^2}}$

$= \frac{(-1)(-0.3)}{\sqrt{16-1}}$

$= 0.077$ m/hr

d) i) The diagonals AC, AF, FC are of equal length so $\triangle ACF$ is equilateral, hence all angles are 60°.

ii) $FA^2 = AB^2 + BF^2$

$= 2^2+2^2$

$FA^2 = 8$

$FA = 2\sqrt{2}$

$FA = AC$ and $AO = \frac{AC}{2}$

$\therefore FO^2 = AF^2 - AO^2$

$= 8 - \sqrt{2}^2$

$= 6$

$\therefore FO = \sqrt{6}$ m.

iii) $XO = 1$ let $\theta = \angle AFO$

$\tan\theta = \frac{XO}{FO} = \frac{1}{\sqrt{6}}$

$\theta = \tan^{-1}\frac{1}{\sqrt{6}}$

$\theta = 22.20765..$

$\angle XFY = 44°$

④ a) <u>Step 1</u>: When $n=3$, $lhs = 1-\frac{2}{3} = \frac{1}{3}$

$rhs = \frac{2}{3(3-1)} = \frac{1}{3}$

so its true for $n=3$

<u>Step 2</u>: Assume its true when $n=k$, so we assume that

$(1-\frac{2}{3})(1-\frac{2}{4})\cdots(1-\frac{2}{k}) = \frac{2}{k(k-1)}$

<u>Step 3</u>: Prove it true when $n=k+1$, so prove that

$(1-2/3)(1-2/4)\cdots(1-\frac{2}{k+1}) = \frac{2}{(k+1)k}$

Start,

$lhs = (1-\frac{2}{3})(1-\frac{2}{4})\cdots(1-\frac{2}{k})(1-\frac{2}{k+1})$

$= \frac{2}{k(k-1)}\left(1-\frac{2}{k+1}\right)$ by step 3

$= \frac{2}{k(k-1)}\ \frac{k-1}{k+1} = \frac{2}{k(k+1)} = rhs.$

so its true when $n=k+1$.

<u>Step 4</u>. Since it is true when $n=3$ and it was shown its true when $n=k+1$ when true for $n=k$, then it holds true for all integers $n \geq 3$.

ct'd

④ b) (i) Tangents at P, Q are

$y = px - ap^2$

$y = qx - aq^2$ resp.

Solve, $px - ap^2 = qx - aq^2$

$x(p-q) = a(p^2 - q^2)$

$x = a(p+q)$ $p \neq q$

$\therefore y = p^2 a + apq - ap^2$

$= apq$

$\therefore R\,(a(p+q), apq)$

as required.

(ii)

gradients,

$m_{OP} = ap^2/2ap = p/2$

$m_{OQ} = q/2$

So $m_{OP} \cdot m_{OQ} = -1$ as $\angle POQ = 90°$

$\therefore \; p/2 \cdot q/2 = -1$

$pq = -4.$

$\therefore R\left(a\left(p - \frac{4}{p}\right), -4a\right)$

$x = a(p - 4/p)$ $p \in \mathbb{R}$

$y = -4a$

Hence x takes all real values while y is constant at $-4a$

$\therefore$ the locus of R is a straight line $y = -4a$

c) (i) $P(\text{no prizes in 7 wks}) = \left(\frac{9}{10}\right)^7$

$P\left(\text{winning at least one prize}\right) = 1 - \left(\frac{9}{10}\right)^7$

(ii) $p = 0{\cdot}1$, $q = 0{\cdot}9$

trials is $n = 20$

The probability that Katie wins exactly twice is

$P\left(\text{win exactly 2}\right) = {}^{20}C_2\,(0{\cdot}1)^2(0{\cdot}9)^{18} \doteqdot 0{\cdot}285$

and of winning exactly once,

$P\left(\text{win exactly one}\right) = {}^{20}C_1\,(0{\cdot}1)(0{\cdot}9)^{19} \doteqdot 0{\cdot}270$

So the probability of winning exactly twice is greater than winning exactly once.

(iii) ${}^nC_3\,(0{\cdot}1)^3(0{\cdot}9)^{n-3} > {}^nC_2\,(0{\cdot}1)^2(0{\cdot}9)^{n-2}$

${}^nC_3\; 0{\cdot}1 > {}^nC_2\; 0{\cdot}9$

${}^nC_3 > 9\; {}^nC_2$

Find the least integer $n > 0$ so

${}^nC_3 - 9\; {}^nC_2 > 0$

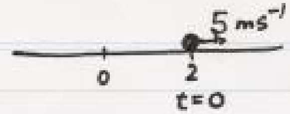
$\frac{\cancel{n(n-1)}(n-2)}{3 \cdot \cancel{2}} - 9\,\frac{\cancel{n(n-1)}}{\cancel{2}} > 0$

$n - 2 > 27$

$n > 29$

$\therefore$ $\underline{n = 30 \text{ weeks}}$

⑤ a)



$\ddot{x} = 2x^3 + 2x$

(i) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = 2x^3 + 2x$

$\frac{v^2}{2} = \frac{x^4}{2} + x^2 + c$

$v^2 = x^4 + 2x^2 + c'$

$25 = 16 + 8 + c'$

$c' = 1$

$v^2 = x^4 + 2x^2 + 1$

$v^2 = (x^2 + 1)^2$

$\dot{x} = x^2 + 1$ where the positive square root was taken to match the given information.

(ii) $\frac{dx}{dt} = x^2 + 1$

$\int \frac{dx}{x^2+1} = \int dt$

$\tan^{-1} x = t + c$

$c = \tan^{-1} 2$

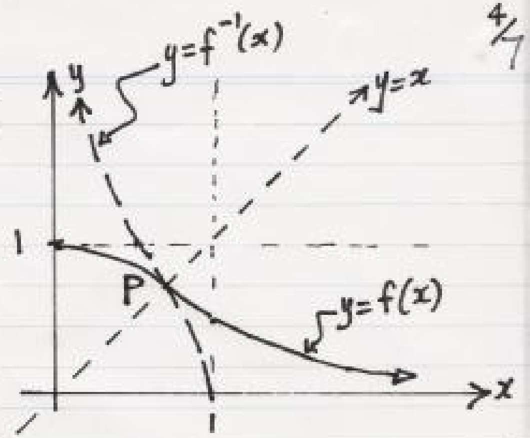
$\tan^{-1} x - \tan^{-1} 2 = t$

$\frac{x - 2}{1 + 2x} = \tan t$

Finally,

$x = \frac{2 + \tan t}{1 - 2\tan t}$

b) (i)



[The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$.]

(ii) Domain of $f^{-1}(x)$ is $0 < x \leq 1$

(iii) inverse relation is $x = \frac{1}{1+y^2}$

$1 + y^2 = \frac{1}{x}$

$y^2 = \frac{1}{x} - 1$

$y = \sqrt{\frac{1}{x} - 1}$ inverse function.

(iv) At P, $f(x) = x$ since P lies on $y = x$.

$\therefore \frac{1}{1+x^2} = x$

$\therefore x^3 + x - 1 = 0$

(v) $x_1 = 0{\cdot}5$ let $F(x) = x^3 + x - 1$

$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$

$= 0{\cdot}5 - \frac{[0{\cdot}5^3 + 0{\cdot}5 - 1]}{[3 \times 0{\cdot}5^2 + 1]}$

$\underline{x_2 = 0{\cdot}71}$

⑥ a) i) $\angle DBF + \angle ABD = \pi$ } (straight angles)
$\angle ACD + \angle ACF = \pi$
$\angle ABD = \angle ACD$ (angles on same chord)

$\therefore \angle DBF = \angle ACF$
$\angle DBF + \angle ACF = \pi$ (opposite vertices of cyclic quadrilateral)
$\therefore 2\angle DBF = \pi$
$\angle DBF = \frac{\pi}{2}$.

(ii) Since $\angle ABD$ is a right angle, AD must be a diameter
$\therefore AD = 2r$

b) (i) $y = 0 \Rightarrow$
$t\left(v\sin\theta - \frac{gt}{2}\right) = 0$
$t > 0 \therefore t = \dfrac{2v\sin\theta}{g}$

At this time,
$x = v\cos\theta .\ 2v\sin\theta / g$
$x = \dfrac{v^2\sin 2\theta}{g}$

(ii) from i)
$\dfrac{v^2}{g}\sin(2\times 15^\circ) = 40$
$v^2 = \dfrac{40g}{\sin 30^\circ}$
$\therefore v^2 = 80g$.

(iii) eliminate t from parametric equations; $t = \frac{x}{v}\sec\theta$

$\therefore y = v\sin\theta \dfrac{x}{v\cos\theta} - \frac{1}{2}g\dfrac{x^2}{v^2}\sec^2\theta$

(iii) ct'd. using ii) we have
$y = x\tan\theta - \dfrac{x^2\sec^2\theta}{160}$ as required.

(iv) from iii)
$\cancel{20} = \cancel{40}\tan\theta - \dfrac{\cancel{40}\times\cancel{40}\times\sec^2\theta}{\cancel{160}}$

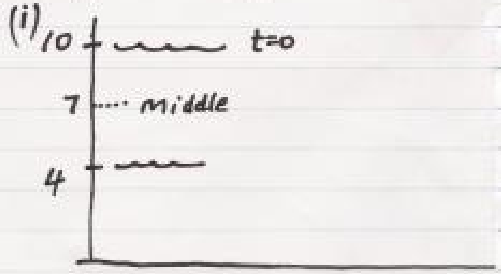
$2 = 4\tan\theta - (1 + \tan^2\theta)$
$\therefore \tan^2\theta - 4\tan\theta + 3 = 0$ as req'd.

(v) We wish to solve $0 < y < 20$
We do this in two parts (two inequalities)
First $y = 40\tan\theta - 10(1 + \tan^2\theta)$
Find range of θ so $y > 0$
$y = -10(\tan\theta - 2 + \sqrt{3})(\tan\theta - 2 - \sqrt{3}) > 0$
$\therefore 2 - \sqrt{3} < \tan\theta < 2 + \sqrt{3}$
$\tan^{-1}(2-\sqrt{3}) < \theta < \tan^{-1}(2+\sqrt{3})$
$15^\circ < \theta < 75^\circ$

Next, find range of θ so $y < 20$
$y = 40\tan\theta - 10 - 10\tan^2\theta < 20$
$\therefore \tan^2\theta - 4\tan\theta + 3 > 0$
$(\tan\theta - 1)(\tan\theta - 3) > 0$
$\therefore \tan\theta < 1$ OR $\tan\theta > 3$
$\theta < 45^\circ$ OR $\theta > 71^\circ 34'$

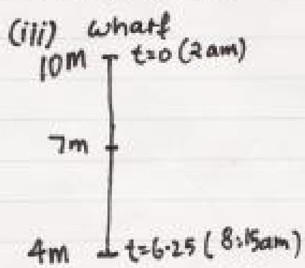
Hence to hit the front of the wall we must have
$15^\circ < \theta < 45^\circ$ OR $71^\circ 34' < \theta < 75^\circ$

⑦ a) $T = 12.5$ hrs

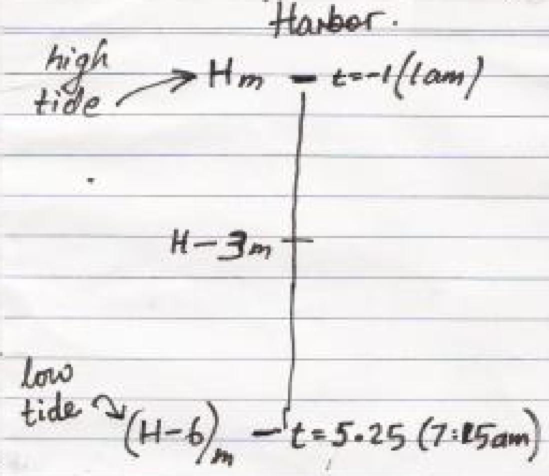
(i)


When $t = 0$ $y = 10$
Centre of motion is $y = 7$.
$T = \frac{2\pi}{n}$, amplitude $a = 10 - 7 =$
$y - 7 = a\cos(nt + \alpha)$
$n = 2\pi/T = 2\pi/12.5 = \frac{4\pi}{25}$
$10 - 7 = 3\cos(0 + \alpha)$
$\cos\alpha = 1 \quad \therefore \alpha = 0$
$\therefore y = 7 + 3\cos\dfrac{4\pi t}{25}$

(ii) $7 + 3\cos\dfrac{4\pi t}{25} \leq 8.5$
$\cos\dfrac{4\pi t}{25} \leq \frac{1}{2}$
$\therefore \dfrac{4\pi t}{25} = \dfrac{\pi}{3}$ earliest value of t.
$t = 25/12 = 2$ hrs 5 min
this corresponds to time on clock of 2 am + 2hrs 5min
$= 4{:}05$ am.

(iii) wharf


$y = 7 + 3\cos\left(\frac{4\pi t}{25}\right)$

Harbor.


$y = H - 3 + 3\cos\frac{4\pi}{25}(t+1)$

$\begin{cases} y(-1) = H \\ y(5.25) = H - 6 \end{cases}$

must have $y \geq H - 6 + 2 = H - 4$.

$\therefore \cancel{H} - 3 + 3\cos\frac{4\pi}{25}(t+1) \geq \cancel{H} - 4$
$\cos\frac{4\pi}{25}(t+1) \geq -\frac{1}{3}$
$\frac{4\pi}{25}(t_0 + 1) = 1.910633236$
$t_0 = 2.8$

This corresponds to the time 4:48 am.
So the latest the ship can leave the wharf is 4:28 am.

7 b) (i) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$

$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \dots + a + 1)$

let $a = x+1$

$\therefore (x+1)^n - 1 = x\left((1+x)^{n-1} + (1+x)^{n-2} + \dots + (1+x) + 1\right)$.

(ii) Consider $(1+x)^m = \sum_{r=1}^{m} \binom{m}{r} x^r$

The coefficient of x^k is $\binom{m}{k}$ $\quad \dots \quad *$

The coefficient of x^k in

$x\left[(1+x)^{n-1} + (1+x)^{n-2} + \dots + (1+x) + 1\right]$

$= x.(1+x)^{n-1} + x(1+x)^{n-2} + \dots + x(1+x) + x$

is $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \dots + \binom{k-1}{k-1}$

↑ coefft of x^{k-1} in $(1+x)^{n-1}$

Hence $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \dots + \binom{k-1}{k-1} = \binom{n}{k}$. as reqd.

(iii) lhs $= n\binom{n-1}{k} = \dfrac{n\,(n-1)!}{k!\,(n-1-k)!} = \dfrac{n!}{k!\,(n-k-1)!}$

rhs $= (k+1)\binom{n}{k+1} = \dfrac{(k+1)\;n!}{(k+1)!\,(n-k-1)!} = \dfrac{n!}{k!\,(n-k-1)!} =$ lhs.

as required.

(iv) First divide by x, then differentiate

$\therefore (n-1)(1+x)^{n-2} + (n-2)(1+x)^{n-3} + \dots + 2(1+x) + 1 =$

$n(1+x)^{n-1}.x^{-1} - (1+x)^n x^{-2} + x^{-2}$

Coefft of x^{k-1} gives,

$(n-1)\binom{n-2}{k-1} + (n-2)\binom{n-3}{k-1} + \dots + k\binom{k-1}{k-1} = n.\binom{n-1}{k} - \binom{n}{k+1}$

$\overset{\text{by (iii)}}{=} (k+1)\binom{n}{k+1} - \binom{n}{k+1}$

$= k\binom{n}{k+1}$

as required.

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