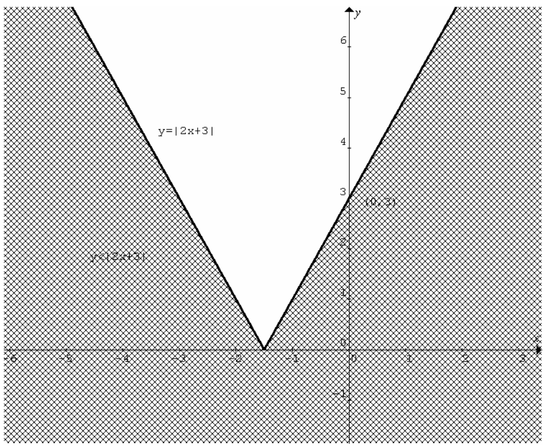
Q1a $\int \frac{1}{x^2+49}dx = \int \frac{1}{7^2+x^2}dx = \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+C$.

Q1b



Q1c For $y=\cos^{-1}\left(\frac{x}{4}\right)$, the domain is $[-4,4]$, the range $[0,\pi]$.

Q1d Let $u=2x^2+1$, $\frac{du}{dx}=4x$, or $x=\frac{1}{4}\frac{du}{dx}$.

$\therefore \int x\left(2x^2+1\right)^{\frac{5}{4}}dx = \int \frac{1}{4}u^{\frac{5}{4}}\frac{du}{dx}dx = \int \frac{1}{4}u^{\frac{5}{4}}du$

$=\frac{1}{9}u^{\frac{9}{4}}+C=\frac{1}{9}\left(2x^2+1\right)^{\frac{9}{4}}+C.$

Q1e $\frac{3\times^{-}1+2x}{5}=1$ and $\frac{3\times 8+2y}{5}=4$, $\therefore x=4$ and $y=-2$.

Q1f Let $\tan\theta=3$ and $\tan\phi=m$, $\therefore$ either $\theta-\phi=45^o$ or $\phi-\theta=45^o$. Hence $\tan(\theta-\phi)=1$ or $\tan(\phi-\theta)=1$.

$\therefore \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{3-m}{1+3m}=1$ or $\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{m-3}{1+3m}=1$.

$\therefore m=\frac{1}{2}$ or -2.

Q2a $\frac{d}{dx}\left(2\sin^{-1}(5x)\right)=5\times\frac{2}{\sqrt{1-(5x)^2}}=\frac{10}{\sqrt{1-25x^2}}$.

Q2b The term independent of x in the expansion of $\left(2x-\frac{1}{x^2}\right)^{12}$ is ${}^{12}C_4(2x)^8\left(-\frac{1}{x^2}\right)^4=126720$.

Q2ci Apply the product rule to find $\frac{d}{dx}\left(e^{3x}(\cos x-3\sin x)\right)$

$=e^{3x}(-\sin x-3\cos x)+3e^{3x}(\cos x-3\sin x)$

$=-10e^{3x}\sin x$.

Q2cii $\int -10e^{3x}\sin x dx=e^{3x}(\cos x-3\sin x)+D$,

$\therefore \int e^{3x}\sin x dx=-\frac{1}{10}e^{3x}(\cos x-3\sin x)+C$.

Q2di $T=3+Ae^{-kt}$, $\therefore \frac{dT}{dt}=-kAe^{-kt}$ and $-k(T-3)=-kAe^{-kt}$.

$\therefore T=3+Ae^{-kt}$ satisfies the differential equation.

Q2dii At $t=0$, $T=25$. At $t=10$, $T=11$. Substitute into $T=3+Ae^{-kt}$ to obtain $25=3+A$, $\therefore A=22$ and $11=3+Ae^{-10k}$, i.e. $\frac{4}{11}=e^{-10k}$, or $k=\frac{1}{10}\log_e\left(\frac{11}{4}\right)$.

At $t=15$, $T=3+22e^{-15k}=7.8^o$ C.

Q3ai $g(0.7)=-0.041$, $g(0.9)=0.168$. $\therefore g(x)$ has a zero between 0.7 and 0.9.

Q3aii Halving the interval, $x=\frac{0.7+0.9}{2}=0.8$,

$g(0.8)=0.052$, $\therefore g(x)$ has a zero between 0.7 and 0.8.

Halving the interval, $x=\frac{0.7+0.8}{2}=0.75$,

$g(0.75)=0.003$, $\therefore g(x)$ has a zero between 0.7 and 0.75.

To one decimal place, $\therefore g(x)$ has a zero at 0.7.

Q3bi $\sin(5x+4x)+\sin(5x-4x)$

$=\sin 5x\cos 4x+\cos 5x\sin 4x+\sin 5x\cos 4x-\cos 5x\sin 4x$

$=2\sin 5x\cos 4x$.

Q3bii $\int \sin 5x\cos 4x dx=\int \frac{1}{2}(\sin 9x+\sin x)dx$

$=-\frac{1}{18}\cos 9x-\frac{1}{2}\cos x+C$.

Q3c $f(x)=x^2+5x$, $f(x+h)=(x+h)^2+5(x+h)$,

$f'(x)=\lim_{h\to 0}\frac{(x+h)^2+5(x+h)-x^2-5x}{h}=\lim_{h\to 0}(2x+h+5)=2x+5$

Q3di $EB=7-4=3$, $EC=\ell-x$. $AE\times EB=DE\times EC$,

$\therefore 12=x(\ell-x)$, or $x^2-\ell x+12=0$.

Q3dii For this quadratic equation to have real x for its solutions, $\Delta\ge 0$, i.e. $\ell^2-4(1)(12)\ge 0$, $\ell^2\ge 48$. Since $\ell>0$, $\therefore \ell\ge\sqrt{48}$. Hence the shortest chord has length $\sqrt{48}=4\sqrt{3}$.

Q4a Let $u = \sin x$, when $x = 0,\ u = 0$; when $x = \frac{\pi}{4},\ u = \frac{1}{\sqrt{2}}$.

$\frac{du}{dx} = \cos x.$

$$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 \frac{du}{dx} dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 du = \left[\frac{u^3}{3}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{12}.$$

Q4b $\cos ec\theta + \cot\theta = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta}$

$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}.$

Q4ci Solve simultaneously to find R,

$x + py = 2ap + ap^3$ …….(1)

$x + qy = 2aq + aq^3$ …….(2) where $p \neq q$.

$(2) - (1)$, $qy - py = 2aq - 2ap + aq^3 - ap^3$,

$(q-p)y = 2a(q-p) + a(q^3 - p^3)$,

$(q-p)y = 2a(q-p) + a(q-p)(q^2 + qp + p^2)$,

$(q-p)y = a(q-p)(2 + q^2 + qp + p^2)$.

Hence $y = a(p^2 + pq + q^2 + 2)$……(3)

Substitute (3) into (1), $x = 2ap + ap^3 - py$

$= 2ap + ap^3 - pa(p^2 + pq + q^2 + 2) = -apq(p+q)$.

$\therefore R$ is $(-apq[p+q], a[p^2 + pq + q^2 + 2])$.

Q4cii $(0, a)$ satisfies $y = \frac{1}{2}(p+q)x - apq$, $\therefore a = -apq$ and $pq = -1$.

Q4ciii Since $pq = -1$, $\therefore R$ is $(a[p+q], a[p^2 + pq + q^2 + 2])$.

$\therefore x = a(p+q)$ or $\frac{x}{a} = (p+q)$…….(1) and

$y = a(p^2 + pq + q^2 + 2) = a(p^2 + 2pq + q^2 - pq + 2)$

$= a((p+q)^2 + 1 + 2)$,

i.e. $y = a((p+q)^2 + 3)$………(2)

Substitute (1) into (2), $y = a\left(\left(\frac{x}{a}\right)^2 + 3\right)$, $\therefore y = \frac{1}{a}x^2 + 3a.$

Q4d For $n = 2$, $4^n - 1 - 7n = 4^2 - 1 - 7 \times 2 = 1 > 0$.

Assume $4^n - 1 - 7n > 0$ is true for $n = k > 2$,

i.e. $4^k - 1 - 7k > 0$, then for $n = k + 1$,

$4^{k+1} - 1 - 7(k+1) = 4 \times 4^k - 1 - 7k - 7$

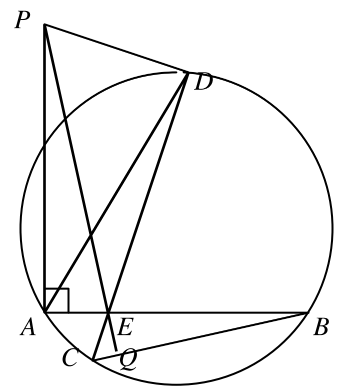
$= 4(4^k - 1 - 7k) + 21k - 4 > 0$. Hence $4^n - 1 - 7n > 0$ for all $n \geq 2$.

Q5a $V = \int_0^{\frac{\pi}{8}} \pi \sin^2 2x dx = \int_0^{\frac{\pi}{8}} \frac{\pi}{2}(1 - \cos 4x) dx = \left[\frac{\pi}{2}\left(x - \frac{\sin 4x}{4}\right)\right]_0^{\frac{\pi}{8}}$

$= \frac{\pi}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi(\pi - 2)}{16}.$

Q5bi Quadrilateral $DPAE$ is cyclic because the sum of the opposite angles is 180^o.

Q5bii $\angle PDC = 90^o$



$\angle APE = \angle ADE$, because they are subtended by the same arc AE of the circle that passes through the vertices D, P, A and E. The circle is not shown in the above diagram.

$\angle ADC = \angle ABC$, because they are subtended by the same arc AC of the circle shown above. Since $\angle ADE$ and $\angle ADC$ are the same angle, $\therefore \angle APE = \angle ABC$.

Q5biii Consider ΔAPE and ΔQBE.

Since $\angle APE = \angle ABC = \angle QBE$ and $\angle AEP = \angle QEB$ (vertically opposite angles are equal), $\therefore$ the third pair of angles must be the same, i.e. $\angle EQB = \angle EAP = 90^o$. $\therefore PQ \perp BC$.

Q5ci Let $\sqrt{3}\sin 3t - \cos 3t = R\sin(3t - \alpha)$, re-express the RHS to obtain $\sqrt{3}\sin 3t - \cos 3t = R\sin 3t \cos\alpha - R\cos 3t \sin\alpha$.

Compare the two sides, $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$.

Hence $\tan\alpha = \frac{1}{\sqrt{3}}$ and $R^2\cos^2\alpha + R^2\sin^2\alpha = 4$.

$\therefore \alpha = \frac{\pi}{6}$ and $R = 2$. $\therefore \sqrt{3}\sin 3t - \cos 3t = 2\sin\left(3t - \frac{\pi}{6}\right)$.

Q5cii $x = 5 + \sqrt{3}\sin 3t - \cos 3t = 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$, hence the particle oscillates about $x = 5$, the centre of motion, with an amplitude of 2 units.

Q5ciii Maximum speed occurs when the particle passes through the centre of motion, where $\sin\left(3t - \frac{\pi}{6}\right) = 0$,

$3t - \frac{\pi}{6} = 0,\ t = \frac{\pi}{18}$.

Q6ai Binomial distribution: $n=5,\ p=\frac{2}{3}$.

To earn one point Megan needs to pick 3 or more.

$$\Pr(X\ge 3)={}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2+{}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1+{}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0=0.7901$$

Q6aii Binomial distribution: $n=18,\ p=0.7901$.

$\Pr(X=18)=0.7901^{18}=0.01$.

Q6aiii Binomial distribution: $n=18,\ p=0.7901,\ q=0.2099$.

$\Pr(X\le 16)=1-\Pr(X=17)-\Pr(X=18)$

$=1-{}^{18}C_{17}(0.7901)^{17}(0.2099)^1-0.01=0.92$.

Q6bi Maximum height is reached when $\frac{dy}{dt}=0$.

$y=-4.9t^2+200t+5000,\ \frac{dy}{dt}=-9.8t+200=0,\ \therefore t=20.4$ s and $y=7040.8$ m.

Q6bii $x=200t,\ \therefore \frac{dx}{dt}=200$.

Descending slope is $\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{-9.8t+200}{200}$.

At 45^o, $\frac{dy}{dx}=\tan(-45^o)=-1,\ \therefore \frac{-9.8t+200}{200}=-1,\ t=40.8$ s.

At 60^o, $\frac{dy}{dx}=\tan(-60^o)=-\sqrt{3}$,

$\therefore \frac{-9.8t+200}{200}=-\sqrt{3},\ t=55.8$ s.

Earliest time is 40.8 s and the latest time is 55.8 s.

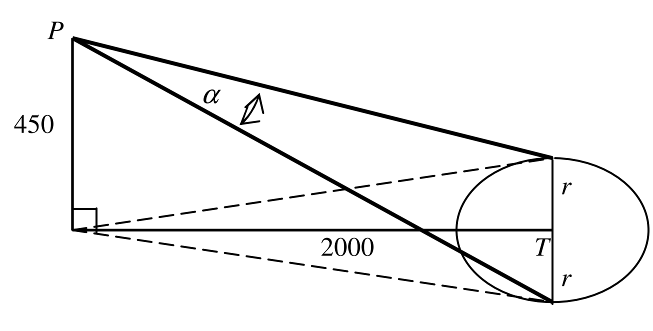
Q6biii The latest time is when the speed = 350 ms^{-1},

i.e. $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=350,\ \therefore 200^2+(200-9.8t)^2=122500,$

$t=49.7$ s.

Q7ai $PT=\sqrt{450^2+2000^2}=2050$ m

$\frac{r}{2050}=\tan(0.05),\ \therefore r=102.6$ m.



Q7aii $r=2050\tan\left(\frac{\alpha}{2}\right),\ \therefore \frac{dr}{d\alpha}=1025\sec^2\left(\frac{\alpha}{2}\right)$.

Given $\frac{d\alpha}{dt}=0.02$ and $\alpha=0.1$,

$\therefore \frac{dr}{dt}=\frac{dr}{d\alpha}\times\frac{d\alpha}{dt}=20.5\sec^2(0.05)=20.6$ m per hour.

Q7bi $f(x)=Ax^3-Ax+1$, where $A>0$.

$f'(x)=3Ax^2-A.\ \therefore f'\left(\pm\frac{\sqrt{3}}{3}\right)=3A\left(\pm\frac{\sqrt{3}}{3}\right)^2-A=0.$

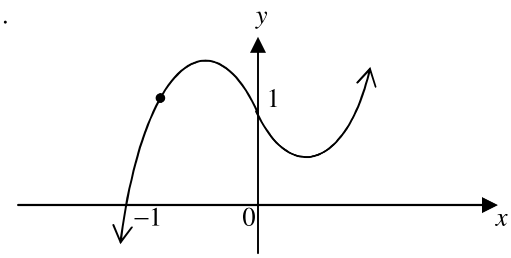
$\therefore f(x)$ has stationary points at $x=\pm\frac{\sqrt{3}}{3}$.

Q7bii At $x=\frac{\sqrt{3}}{3}$, the value of the function is a minimum,

$y=f\left(\frac{\sqrt{3}}{3}\right)=A\left(\frac{\sqrt{3}}{3}\right)^3-A\left(\frac{\sqrt{3}}{3}\right)+1=-\frac{2A}{3\sqrt{3}}+1.$

For the local minimum to be a positive value, $-\frac{2A}{3\sqrt{3}}+1>0$,

$\therefore A<\frac{3\sqrt{3}}{2}$.



$\therefore f(x)$ has exactly one zero when $A<\frac{3\sqrt{3}}{2}$.

Q7biii Since $f(-1)=1$ and $f'(-1)=2A>0$, given $A>0$, the only zero must be at $x<-1$. $\therefore f(x)$ does not have a zero in the interval $-1\le x\le 1$ when $0<A<\frac{3\sqrt{3}}{2}$.

Q7biv $g(\theta)=2\cos\theta+\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

$g'(\theta)=-2\sin\theta+\sec^2\theta=-2\sin\theta+\frac{1}{\cos^2\theta}$

$=\frac{-2\sin\theta\cos^2\theta+1}{\cos^2\theta}=\frac{-2\sin\theta(1-\sin^2\theta)+1}{\cos^2\theta}$

$=\frac{2\sin^3\theta-2\sin\theta+1}{\cos^2\theta}$. Since $0<2<\frac{3\sqrt{3}}{2}$ and $-1<\sin\theta<1$

$\because -\frac{\pi}{2}<\theta<\frac{\pi}{2},\ \therefore 2\sin^3\theta-2\sin\theta+1$ has no zeros. Hence $g(\theta)$ does not have any stationary points.

Q7bv $\therefore g(\theta)$ must be a one-to-one function in the interval $-\frac{\pi}{2}<\theta<\frac{\pi}{2},\ \therefore g(\theta)$ has an inverse function.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.

