

Q1

(a) $\int \frac{dx}{49+x^2} = \frac{1}{7} \tan^{-1} \frac{x}{7} + C$

(b) $\int x^3 \sqrt{x^4+8} dx = \frac{1}{4} \int \sqrt{u} du = \frac{1}{4} \frac{2\sqrt{u}^3}{3} = \frac{1}{6} \sqrt{(x^4+8)^3} + C$

(c) $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times \frac{5}{3} = \frac{5}{3}$

(d) $\frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} - 1$
 $= \frac{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)}{\sin \theta + \cos \theta} - 1$
 $= \sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta - 1$
 $= 1 - \sin \theta \cos \theta - 1 = -\sin \theta \cos \theta \left(= -\frac{1}{2} \sin 2\theta \right)$

(e) Solving $y = x^3$ and $y = 12x + b$ gives $x^3 - 12x - b = 0$.

Let $f(x) = x^3 - 12x - b$.

$f'(x) = 0, \therefore 3x^2 - 12 = 0, \therefore x^2 = 4, \therefore x = \pm 2$.

$f(2) = 2^3 - 12 \times 2 - b = 0, \therefore b = 8 - 24 = -16$.

$f(-2) = (-2)^3 - 12 \times (-2) - b = 0, \therefore b = -8 + 24 = 16$.

$\therefore b = \pm 16$

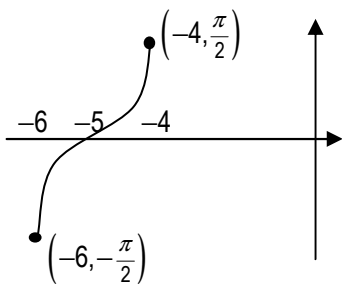
Q2

(a) (i) Domain: $-1 \leq x + 5 \leq 1$ gives $-6 \leq x \leq -4$.

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

(ii) $y' = \frac{1}{\sqrt{1-(x+5)^2}}$. When $x = -5, y' = \frac{1}{\sqrt{1}} = 1$.

(iii)



(b) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$.

Differentiating both sides with respect to x gives

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1}$.

Substituting x by 2 gives

$n3^{n-1} = \binom{n}{1} + 2\binom{n}{2}2 + 3\binom{n}{3}2^2 + \dots + n\binom{n}{n}2^{n-1}$.

The r th term is $r\binom{n}{r}2^{r-1}$.

(c) (i) When $x = 0, y = -apr, \therefore U(0, -apr)$

(ii) $y = px - ap^2$ (1)

$y = qx - aq^2$ (2)

(1) - (2) gives

$(p-q)x - a(p^2 - q^2) = 0$.

$x = \frac{a(p^2 - q^2)}{p-q} = a(p+q)$.

Sub. to (1), $y = pa(p+q) - ap^2 = apq$.

$\therefore T(a(p+q), apq)$.

(iii) Since QR is perpendicular to the y -axis, Q and R are symmetrical about the y -axis, $\therefore q = -r$.

\therefore The y -coordinates of U and T are the same.

$\therefore UT$ is perpendicular to the y -axis.

Q3

(a) $\int_0^{\pi/4} \sin^2 x dx = \int_0^{\pi/4} \frac{1 - \cos 2x}{2} dx$

$= \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}$.

(b) (i) When $x = 1.5, f(1.5) = -0.28 < 0$.

When $x = 2, f(2) = 0.08 > 0$.

\therefore The curves $\ln x$ and x are continuous on the interval $[1.5, 2], \therefore 3 \ln x - x$ is continuous and as it goes from a negative to a positive value it meets the x -axis at least once, \therefore there is (at least) a root between 1.5 and 2.

(ii) $f'(x) = \frac{3}{x} - 1$.

$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{3 \ln 1.5 - 1.5}{\frac{3}{1.5} - 1} = 1.78$ (2 d.p.)

(c) (i) ${}^5P_3 = 60$.

(ii) ${}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5 = 320$.

(d) (i) $\angle QMT + \angle QKT = 180^\circ$.

\therefore $QKTM$ is a cyclic quad (opposite angles are supplementary).

(ii) $\angle KMT = \angle KQT$ (angles subtending the same arc are equal).

(iii) $\angle KQT = \angle PTN$ (the angle between a chord and a tangent is equal to any angle in the alternate segment)

$\therefore \angle KMT = \angle PTN$.

$\therefore KM \parallel PT$ (corresponding angles on parallel lines are equal).

Q4

(a) (i) $\sum \alpha = 1 = -r, \therefore r = -1$.

(ii) $\sum \alpha\beta = \alpha - \alpha - \alpha^2 = s, \therefore s = -\alpha^2$.

$\prod \alpha = -\alpha^2 = -t, \therefore t = \alpha^2$.

$\therefore s + t = 0$

$$(b) \text{ (i) Period} = 5 \text{ s, } 5 = \frac{2\pi}{n}, \therefore n = \frac{2\pi}{5}.$$

\therefore The equation of motion is $x = 18 \cos \frac{2\pi}{5} t$, noting that

when $t = 0, x = 18$.

(ii) When $x = 9$,

$$9 = 18 \cos \frac{2\pi}{5} t$$

$$\cos \frac{2\pi}{5} t = \frac{1}{2}$$

$$\frac{2\pi}{5} t = \frac{\pi}{3}$$

$$t = \frac{5}{6} \text{ s.}$$

$$(c) \text{ (i) } \ddot{x} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 18x^3 + 27x^2 + 9x.$$

$$\frac{1}{2} v^2 = \frac{9x^4}{2} + 9x^3 + \frac{9x^2}{2} + C.$$

When $x = -2, v = -6$,

$$18 = 72 - 72 + 18 + C.$$

$$C = 0.$$

$$\therefore v^2 = 9(x^4 + 2x^3 + x^2) = 9x^2(x+1)^2$$

$$(ii) v = \frac{dx}{dt} = -3x(x+1) \text{ (the minus sign is taken to satisfy}$$

that when $x = -2, v = -6$)

$$-3 \frac{dt}{dx} = \frac{1}{x(x+1)}$$

$$-3t = \int \frac{1}{x(x+1)} dx$$

$$(iii) 3t + C = \ln \left(1 + \frac{1}{x} \right)$$

$$\text{When } t = 0, x = -2, C = \ln \left(1 - \frac{1}{2} \right) = \ln \frac{1}{2}.$$

$$3t = \ln \left(1 + \frac{1}{x} \right) - \ln \frac{1}{2} = \ln \left(2 + \frac{2}{x} \right).$$

$$e^{3t} = 2 + \frac{2}{x}.$$

$$\frac{2}{x} = e^{3t} - 2.$$

$$x = \frac{2}{e^{3t} - 2}.$$

Q5

$$(a) \frac{dy}{dt} = -7e^{-0.7t}$$

$$\text{But } e^{-0.7t} = \frac{y-3}{10}, \therefore \frac{dy}{dt} = -0.7(y-3).$$

$$(b) f'(x) = \frac{e^x}{1+e^x}$$

As $e^x > 0, f'(x) > 0, \therefore f(x)$ is monotonic increasing, $\therefore f(x)$ is 1:1, \therefore It has an inverse.

$$(c) V = \frac{\pi}{3} (3rx^2 - x^3)$$

$$\therefore \frac{dV}{dx} = \frac{\pi}{3} (6rx - 3x^2) = \pi (2rx - x^2).$$

$$\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = \pi (2rx - x^2) \frac{dx}{dt} = k.$$

$$\therefore \frac{dx}{dt} = \frac{k}{\pi (2rx - x^2)} = \frac{k}{\pi x(2r - x)}$$

$$(ii) \frac{dt}{dx} = \frac{\pi}{k} (2rx - x^2).$$

$$t = \frac{\pi}{k} \int (2rx - x^2) dx.$$

Let T_1 be the time to fill up to $\frac{r}{3}$ and T_2 be the time

to fill up to $\frac{2r}{3}$.

$$T_1 = \frac{\pi}{k} \int_0^{\frac{r}{3}} (2rx - x^2) dx = \frac{\pi}{k} \left[rx^2 - \frac{x^3}{3} \right]_0^{\frac{r}{3}}$$

$$= \frac{\pi}{k} \left(\frac{r^3}{9} - \frac{r^3}{81} \right) = \frac{\pi r^3}{k} \times \frac{8}{81}.$$

$$T_2 = \frac{\pi}{k} \int_0^{\frac{2r}{3}} (2rx - x^2) dx = \frac{\pi}{k} \left[rx^2 - \frac{x^3}{3} \right]_0^{\frac{2r}{3}}$$

$$= \frac{\pi}{k} \left(\frac{4r^3}{9} - \frac{8r^3}{81} \right) = \frac{\pi r^3}{k} \times \frac{28}{81}.$$

$$\therefore \frac{T_2}{T_1} = \frac{28}{8} = 3.5.$$

$\therefore T_2$ is 3.5 times T_1 .

$$(d) \text{ (i) } \tan \alpha - \tan \beta = \tan(\alpha - \beta)(1 + \tan \alpha \tan \beta)$$

$$\therefore \tan(n+1)\theta - \tan n\theta = \tan \theta (1 + \tan(n+1)\theta \tan n\theta)$$

$$\text{RHS} = \frac{1}{\tan \theta} \tan \theta (1 + \tan(n+1)\theta \tan n\theta)$$

$$= 1 + \tan(n+1)\theta \tan n\theta$$

$$= \text{LHS.}$$

(ii) When $n = 1$, LHS = $\tan \theta \tan 2\theta$,

From (i), $\tan n\theta \tan(n+1)\theta =$

$$-1 + \cot \theta (\tan(n+1)\theta - \tan n\theta)$$

$$\therefore \tan \theta \tan 2\theta = -1 + \cot \theta (\tan 2\theta - \tan \theta)$$

$$= -1 + \cot \theta \tan 2\theta - 1$$

$$= -2 + \cot \theta \tan 2\theta = \text{RHS.}$$

Assume $\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \dots$

$$+ \tan n\theta \tan(n+1)\theta = -(n+1) + \cot \theta \tan(n+1)\theta.$$

Required to prove that

$$\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \dots + \tan n\theta \tan(n+1)\theta$$

$$+ \tan(n+1)\theta \tan(n+2)\theta = -(n+2) + \cot \theta \tan(n+2)\theta.$$

$$\begin{aligned} \text{LHS} &= -(n+1) + \cot \theta \tan(n+1)\theta + \tan(n+1)\theta \tan(n+2)\theta \\ &= -(n+1) + \cot \theta \tan(n+1)\theta - 1 \\ &\quad + \cot \theta (\tan(n+2)\theta - \tan(n+1)\theta) \\ &= -(n+2) + \cot \theta \tan(n+2)\theta \\ &= \text{RHS}. \end{aligned}$$

\therefore The statement is true for $n+1$.

Since the statement is true for $n=1$, and $n+1$ if true for n , it is true for all $n \geq 1$.

Q6

$$(a) (i) L^2 = (Vt \cos \theta - \alpha)^2 + \left(Vt \sin \theta - \frac{1}{2}gt^2 - Vt + \frac{1}{2}gt^2 \right)^2$$

$$\begin{aligned} &= V^2t^2 \cos^2 \theta - 2\alpha Vt \cos \theta + \alpha^2 + V^2t^2 \sin^2 \theta + V^2t^2 \\ &\quad - 2V^2t^2 \sin \theta \\ &= V^2t^2 (\cos^2 \theta + \sin^2 \theta) + V^2t^2 - 2\alpha Vt \cos \theta + \alpha^2 \\ &\quad - 2V^2t^2 \sin \theta \end{aligned}$$

$$= 2V^2t^2 - 2\alpha Vt \cos \theta + \alpha^2 - 2V^2t^2 \sin \theta$$

$$= 2V^2t^2 (1 - \sin \theta) - 2\alpha Vt \cos \theta + \alpha^2$$

$$(ii) \frac{dL^2}{dt} = 4V^2t(1 - \sin \theta) - 2\alpha V \cos \theta$$

$$\frac{dL^2}{dt} = 0 \text{ when } t = \frac{2\alpha V \cos \theta}{4V^2(1 - \sin \theta)} = \frac{\alpha \cos \theta}{2V(1 - \sin \theta)}$$

$$\frac{d^2L^2}{dt^2} = 4V^2(1 - \sin \theta) > 0, \therefore \text{The distance is}$$

$$\text{minimum when } t = \frac{\alpha \cos \theta}{2V(1 - \sin \theta)}.$$

Substituting to (i)

$$L^2 = 2V^2 \frac{\alpha^2 \cos^2 \theta}{4V^2(1 - \sin \theta)^2} (1 - \sin \theta) - \frac{2\alpha^2 V \cos^2 \theta}{2V(1 - \sin \theta)} + \alpha^2$$

$$= \frac{\alpha^2 \cos^2 \theta}{2(1 - \sin \theta)} - \frac{\alpha^2 \cos^2 \theta}{1 - \sin \theta} + \alpha^2$$

$$= \alpha^2 - \frac{\alpha^2 \cos^2 \theta}{2(1 - \sin \theta)} = \alpha^2 - \frac{\alpha^2(1 - \sin^2 \theta)}{2(1 - \sin \theta)}$$

$$= \alpha^2 - \frac{\alpha^2(1 + \sin \theta)}{2}$$

$$= \frac{\alpha^2(2 - 1 - \sin \theta)}{2} = \frac{\alpha^2(1 - \sin \theta)}{2}$$

$$\therefore \text{The smallest distance is } \alpha \sqrt{\frac{1 - \sin \theta}{2}}.$$

(iii) If particle 1 is ascending then $\dot{y} > 0$.

$$V \sin \theta - gt > 0$$

$$V \sin \theta - g \frac{\alpha \cos \theta}{2V(1 - \sin \theta)} > 0$$

$$2V^2 \sin \theta (1 - \sin \theta) - g\alpha \cos \theta > 0.$$

$$V^2 > \frac{g\alpha \cos \theta}{2 \sin \theta (1 - \sin \theta)}.$$

$$\therefore V > \sqrt{\frac{g\alpha \cos \theta}{2 \sin \theta (1 - \sin \theta)}}.$$

(b) (i) $P(\text{at least 3 not complete}) = P(3 \text{ not complete})$

$$+ P(4 \text{ not complete}) = \binom{4}{3} pq^3 + q^4 = 4pq^3 + q^4$$

(ii) $P(\text{a 4 member team scores point}) = 1 - P(\text{at least 3 not complete}) = 1 - 4pq^3 - q^4$

$$= 1 - 4(1-q)q^3 - q^4$$

$$= 1 - 4q^3 + 4q^4 - q^4 = 1 - 4q^3 + 3q^4$$

(iii) $P(\text{a 2 member team scores point}) = 1 - P(\text{both not complete}) = 1 - q^2$.

$$(iv) 1 - q^2 > 1 - 4q^3 + 3q^4.$$

$$3q^4 - 4q^3 + q^2 < 0.$$

$$3q^2 - 4q + 1 < 0, \text{ dividing by } q^2,$$

$$(3q-1)(q-1) < 0$$

$$\frac{1}{3} < q < 1.$$

Q7

$$(a) A = \frac{1}{2}r^2(2\theta - \sin 2\theta) = \frac{1}{2} \times (2\theta - 2 \sin \theta \cos \theta)$$

$$= r^2(\theta - \sin \theta \cos \theta).$$

$$(b) A = \frac{w^2}{4\theta^2} \left(\theta - \frac{1}{2} \sin 2\theta \right), \text{ since } w = r \times 2\theta, \therefore r = \frac{w}{2\theta}.$$

$$\frac{dA}{d\theta} = \frac{w^2}{4} \left(\frac{(1 - \cos 2\theta)\theta^2 - 2\theta(\theta - \frac{1}{2} \sin 2\theta)}{\theta^4} \right)$$

$$= \frac{w^2}{4} \left(\frac{(1 - \cos 2\theta)\theta - 2\theta + \sin 2\theta}{\theta^3} \right)$$

$$= \frac{w^2}{4} \left(\frac{-\theta - \theta \cos 2\theta + \sin 2\theta}{\theta^3} \right)$$

$$= \frac{w^2}{4} \left(\frac{-\theta - \theta(2 \cos^2 \theta - 1) + 2 \sin \theta \cos \theta}{\theta^3} \right)$$

$$= \frac{w^2}{4} \left(\frac{-2\theta \cos^2 \theta + 2 \sin \theta \cos \theta}{\theta^3} \right)$$

$$= \frac{w^2 \cos \theta}{2} \left(\frac{-\theta \cos \theta + \sin \theta}{\theta^3} \right)$$

$$= \frac{w^2 \cos \theta (\sin \theta - \theta \cos \theta)}{2\theta^3}.$$

(c) $g(\theta) = \sin \theta - \theta \cos \theta$.

$$g'(\theta) = \cos \theta - (\cos \theta - \theta \sin \theta)$$

$$= \theta \sin \theta > 0 \text{ for } 0 < \theta < \pi.$$

$\therefore g(\theta)$ is increasing for $0 < \theta < \pi$.

When $\theta = 0, g(\theta) = 0, \therefore g(\theta) > 0$ for $0 < \theta < \pi$.

(d) $\frac{dA}{d\theta} = 0$ when $\cos \theta = 0$, since $\sin \theta - \theta \cos \theta > 0$

$\therefore \theta = \frac{\pi}{2}$.

\therefore Only 1 value of θ .

(e)

θ	1	$\frac{\pi}{2}$	2
$\cos \theta$	0.54	0	-0.42
$\frac{dA}{d\theta}$	\nearrow	\rightarrow	\searrow

\therefore The area of the gutter is maximum when $\theta = \frac{\pi}{2}$.

Maximum area $= \frac{w^2(\pi - 0)}{2\pi^2} = \frac{w^2}{2\pi}$.