2007 Extension 1 Solution (by Terry Lee)

Q1

(a) $\left(1+\sqrt{5}\right)^3 = 1+3\sqrt{5}+3\left(\sqrt{5}\right)^2+\left(\sqrt{5}\right)^3$

$= 1+3\sqrt{5}+15+5\sqrt{5}$

$= 16+8\sqrt{5}.$

(b) $x = \dfrac{3\times 4+2\times 19}{3+2} = 10$

$y = \dfrac{3\times 5+2\times(-5)}{3+2} = 1.$

(c) $\dfrac{d}{dx}\left(\tan^{-1}(x^4)\right) = \dfrac{4x^3}{1+x^8}.$

(d) For $y = x^3+1, y' = 3x^2$. When $x = 1, m_1 = 3$.

For $x-2y+3=0, m_2 = \dfrac{1}{2}$.

$\tan\alpha = \left|\dfrac{m_1-m_2}{1+m_1m_2}\right| = \left|\dfrac{3-\frac{1}{2}}{1+3\times\frac{1}{2}}\right| = \dfrac{5}{5} = 1, \therefore \alpha = \dfrac{\pi}{4}.$

(e) Let $u = 25-x^2, du = -2x\,dx$.

When $x = 3, u = 16$; When $x = 4, u = 9$.

$\displaystyle\int_3^4 \frac{2x}{\sqrt{25-x^2}}dx = \int_{16}^{9} \frac{-du}{\sqrt{u}}$

$= 2\left[\sqrt{u}\right]_9^{16}$

$= 2(4-3)$

$= 2.$

Q2

(a) $\text{LHS} = \dfrac{1-\dfrac{1-t^2}{1+t^2}}{\dfrac{2t}{1+t^2}}$

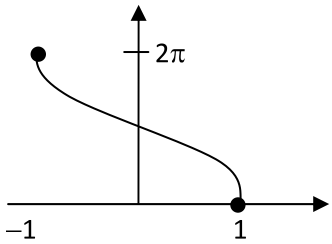
$= \dfrac{1+t^2-1+t^2}{2t}$

$= \dfrac{2t^2}{2t}$

$= t = \text{RHS}.$

(b) (i) See graph.

(ii) $0 \le y \le 2\pi$.



(c) $P(2) = 0$ gives $4+2a+b=0$. (1)

$P(-1) = 18$ gives $18 = 1-a+b, \therefore 17+a-b = 0$ (2)

$(1)+(2)$ gives $21+3a = 0, \therefore a = -7$.

Substituting $a = -7$ to (1) gives $b = 10$.

(d) (i) $a = \dfrac{dv}{dt} = 50\times 0.2e^{-0.2t}$.

When $t = 10, a = 10e^{-2} = 1.4 \text{ m/s}^2$.

(ii) $\dfrac{dx}{dt} = 50\left(1-e^{-0.2t}\right)$

$x = \displaystyle\int_0^{10} 50\left(1-e^{-0.2t}\right) dt$

$= 50\left[t+\dfrac{e^{-0.2t}}{0.2}\right]_0^{10}$

$= 50\left[\left(10+\dfrac{e^{-2}}{0.2}\right)-\dfrac{1}{0.2}\right]$

$= 284$ m.

Q3

(a) $V = \int_0^3 \pi y^2 \, dx$

$= \pi \int_0^3 \frac{1}{9+x^2} \, dx$

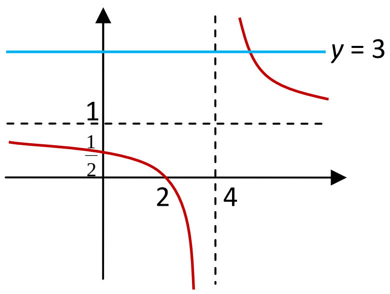
$= \pi \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^3$

$= \frac{\pi}{3} \times \frac{\pi}{4}$

$= \frac{\pi^2}{12}$.

(b) (i) Vertical asymptote $x = 4$.

Horizontal asymptote $y = 1$.



Solving $\frac{x-2}{x-4} = 3$ gives $x - 2 = 3x - 12$

$2x = 10$

$x = 5$.

$\therefore \frac{x-2}{x-4} \le 3$ when $x < 4$ or $x \ge 5$.

(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right), \therefore v^2 = 2\int \ddot{x} \, dx$

$v^2 = 2\int -e^{-2x} \, dx = e^{-2x} + C$.

When $t = 0, x = 0, v = 1, \therefore 1 = 1 + C, \therefore C = 0$.

$\therefore v^2 = e^{-2x}$.

$\therefore v = \pm e^{-x}$.

Initially, $v > 0, \therefore v = e^{-x}$.

(ii) $\frac{dx}{dt} = e^{-x}$

$\frac{dt}{dx} = e^x$.

$t = e^x + C$.

$t = 0, x = 0, \therefore C = -1$.

$\therefore t = e^x - 1$.

$e^x = t + 1$.

$x = \ln(t+1)$.

Q4

(a) (i) $(0.1)^2 = 0.01$

(ii) ${}^{20}C_2 (0.1)^2 (0.9)^{18} = 0.285$

(iii) $1 - \Pr(x = 0, 1, 2)$

$= 1 - 0.9^{20} - {}^{20}C_1 (0.1)(0.9)^{19} - {}^{20}C_2 (0.1)^2 (0.9)^{18}$

$= 0.32$.

(b) Let $n = 1, 7 + 5 = 12$, which is divisible by 12,

$\therefore$ The statement is true for $n = 1$.

Assume the statement is true for $n = k$,

i.e. $7^{2n-1} + 5 = 12M$, where M is an integer.

$\therefore 7^{2n-1} = 12M - 5$

$\therefore 7^{2n+1} + 5 = 49\left(7^{2n-1}\right) + 5$

$= 49(12M - 5) + 5$

$= 588M - 240$

$= 12(49M - 20)$, which is a multiple of 12.

$\therefore$ The statement is true for all $n = k + 1$.

$\therefore$ The statement is true for all $n \ge 1$.

(c) (i) $\angle QXB = \angle DXP$ (vertically opposite)

$\angle DXP = \angle XAP$ (both $= 90° - \angle XDP$, angle sum in Δ)

$\angle XAP = \angle QBX$ (angles subtending the same arc are equal)

$\therefore \angle QXB = \angle QBX$.

(ii) From (i), ΔBXQ is isosceles, $\therefore QB = QX$ (base angles in isosceles Δ).

Similarly, $\angle QXC = \angle QCX, \therefore \Delta CXQ$ is isosceles,

$\therefore QC = QX$.

$\therefore QB = QC$.

$\therefore Q$ bisects BC.

Q5

(a) (i) Area $(\Delta OPT) = \frac{1}{2} OP.TP$

$= \frac{1}{2} r^2 \tan\theta$, since $TP = r\tan\theta$.

Area (sector OPQ) $= \frac{1}{2} r^2 \theta$.

Area $(\Delta OPT) = 2 \times$ Area (sector OPQ),

$\therefore \frac{1}{2} r^2 \tan\theta = 2 \times \frac{1}{2} r^2 \theta.$

$\therefore \tan\theta = 2\theta.$

(ii) Let $f(x) = 2\theta - \tan\theta$.

$f'(x) = 2 - \sec^2\theta.$

By Newton's method, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 1.15 - \frac{2\times 1.15 - \tan 1.15}{2 - \sec^2 1.15}$

$= 1.1664.$

(b) Group the four children together, there are 3 groups,

$\therefore$ 3!4! ways to arrange so that the children stay together.

$\therefore$ Pr(the 4 children stay together) $= \frac{3!4!}{6!} = \frac{1}{5}$.

(c) Let $= u, = v$,

$\sin^{-1} x + \frac{1}{2}\cos^{-1} y = \frac{\pi}{3}$ (1)

$3\sin^{-1} x - \frac{1}{2}\cos^{-1} y = \frac{2\pi}{3}$ (2)

(1) + (2) gives $4\sin^{-1} x = \pi$.

$\therefore \sin^{-1} x = \frac{\pi}{4}, \therefore x = \frac{\sqrt{2}}{2}.$

Substituting $\sin^{-1} x = \frac{\pi}{4}$ into (1),

$\frac{1}{2}\cos^{-1} y = \frac{\pi}{3} - \frac{\pi}{4}$

$\therefore \cos^{-1} y = \frac{\pi}{6}, \therefore y = \frac{\sqrt{3}}{2}.$

(d) (i) Substituting $(2aq, aq^2)$ into the equation of PQ,

$2aq + p(aq^2) - 2ap - ap^3 = 0$

$2q + pq^2 - 2p - p^3 = 0$

$2(p-q) + p(q^2 - p^2) = 0$

$2(p-q) + p(p-q)(p+q) = 0$

$2 + p(p+q) = 0$, since $p \ne q$.

$2 + p^2 + pq = 0.$

(ii) If $OP \perp OQ$ then $\frac{ap^2}{2ap} \times \frac{aq^2}{2aq} = -1, \therefore pq = -4.$

Substituting to part (i), $2 + p^2 - 4 = 0$.

$\therefore p^2 = 2.$

Q6

(a) (i) $x = 3 + \sqrt{3}\sin 2t - \cos 2t$

$\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$

$\ddot{x} = -4\sqrt{3}\sin 2t + 4\cos 2t$

$= -4\left(\sqrt{3}\sin 2t - \cos 2t\right)$

$= -4(x-3).$

(ii) Period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ s.

(iii) $\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$

$= 4\left(\frac{\sqrt{3}}{2}\cos 2t + \frac{1}{2}\sin 2t\right)$

$= 4\left(\cos 2t \cos\frac{\pi}{6} + \sin 2t \sin\frac{\pi}{6}\right)$

$= 4\cos\left(2t - \frac{\pi}{6}\right).$

(iv) When $\dot{x} = \pm 2$,

$\pm 2 = 4\cos\left(2t - \frac{\pi}{6}\right)$

$\cos\left(2t - \frac{\pi}{6}\right) = \pm\frac{1}{2}$

$2t - \frac{\pi}{6} = \pm\frac{\pi}{3} + k2\pi, \pm\frac{2\pi}{3} + k2\pi.$

$2t = \pm\frac{\pi}{3} + \frac{\pi}{6} + k2\pi, \pm\frac{2\pi}{3} + \frac{\pi}{6} + k2\pi.$

$t = \pm\frac{\pi}{6} + \frac{\pi}{12} + k\pi, \pm\frac{\pi}{3} + \frac{\pi}{12} + k\pi.$

For $0 \le t \le \pi, t = \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}.$

(b) (i) $f'(x) = e^x + e^{-x} > 0$ always, $\therefore f(x)$ is increasing.

(ii) $f^{-1}: x = e^y - e^{-y}$ (1)

$x = e^y - \frac{1}{e^y}.$

$xe^y = e^{2y} - 1.$

$e^{2y} - xe^y - 1 = 0$

$e^y = \frac{x \pm \sqrt{x^2+4}}{2}.$

Since $e^y > 0$ for all real $x, e^y = \frac{x + \sqrt{x^2+4}}{2}.$

$\therefore y = \ln\frac{x + \sqrt{x^2+4}}{2}.$ (2)

(iii) Substituting $x = 5$ in (1) gives the same answer as substituting $x = 5$ in (2).

$\therefore$ Solving $e^x - e^{-x} = 5$ gives $x = \ln\frac{5+\sqrt{29}}{2} = 1.65$

Q7

(a) (i) $y = kx^n, \therefore y' = nkx^{n-1}$.

When $x = a, m_1 = nka^{n-1}$.

$y = \ln x, y' = \frac{1}{x}$.

When $x = a, m_2 = \frac{1}{a}$.

If these two gradients are equal, $nka^{n-1} = \frac{1}{a}$.

$\therefore nka^n = 1$.

$\therefore a^n = \frac{1}{nk}$. (1)

(ii) When $x = a, ka^n = \ln a$.

$k \times \frac{1}{nk} = \ln a$, on substituting $a^n = \frac{1}{nk}$.

$\ln a = \frac{1}{n}$.

$a = e^{\frac{1}{n}}$

$a^n = e$.

$\therefore$ (1) becomes $e = \frac{1}{nk}, \therefore k = \frac{1}{ne}$.

(b) (i) $t = \frac{x}{14\cos\theta}$.

$$y = 14\left(\frac{x}{14\cos\theta}\right)\sin\theta - 4.9\left(\frac{x}{14\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{1}{40}x^2\sec^2\theta$$

$$= x\tan\theta - \frac{1}{40}x^2\left(1+\tan^2\theta\right)$$

$$= mx - \left(\frac{1+m^2}{40}\right)x^2, \text{ by letting } \tan\theta = m.$$

(ii) When $x = 10, y = h$,

$$h = 10m - \left(\frac{1+m^2}{40}\right)100$$

$h = 10m - 2.5(1+m^2)$.

$2.5m^2 - 10m + 2.5 + h = 0$.

$$m = \frac{10 \pm \sqrt{100 - 10(2.5+h)}}{5}$$

$$= \frac{10 \pm \sqrt{75 - 10h}}{5}$$

$$= \frac{10 \pm 5\sqrt{3 - 0.4h}}{5}$$

$$= 2 \pm \sqrt{3 - 0.4h}.$$

Since $3 - 0.4h \geq 0, h \leq \frac{3}{0.4} = 7.5$ m.

(iii) Given $m = 2 \pm \sqrt{3 - 0.4h}$.

When $h = 5.9, m = 2 \pm \sqrt{3 - 0.4 \times 5.9} = 2.8$ or 1.2.

When $h = 3.9, m = 2 \pm \sqrt{3 - 0.4 \times 3.9} = 0.8$ or 3.2.

$\therefore$ One interval is $2.8 \leq m \leq 3.2$, and the other interval is $0.8 \leq m \leq 1.2$.

(iv) When $y = 0, mx - \left(\frac{1+m^2}{40}\right)x^2 = 0$

$$x\left(m - \frac{1+m^2}{40}x\right) = 0$$

$x = \frac{40m}{1+m^2}$ m.

When $m = 2.8, x = 12.7$ m

When $m = 3.2, x = 11.4$ m

The width of this interval is $12.7 - 11.4 = 1.3$ m.

When $m = 1.2, x = 19.7$ m

When $m = 0.8, x = 19.5$ m

But m is the gradient of the angle of projection, so when $0.8 \leq m \leq 1.2$, the maximum range is obtained when $\theta = 45°$ (i.e. $m = 1$), $\therefore$ When $m = 1, x = 20$ m.

The width of this interval is $20 - 19.5 = 0.5$ m.