Extension 1 2009 Solution

Q1

(a) $8x^3+27=(2x+3)(4x^2-6x+9)$.

(b) $x>3$.

(c) $\lim_{x\to 0}\frac{\sin 2x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2$.

(d) $\frac{x+3}{2x}>1$.

$(x+3)x>2x^2$

$x^2+3x-2x^2>0$

$-x^2+3x>0.$

$x(-x+3)>0.$

$\therefore 0<x<3.$

(e) $\frac{d}{dx}\left(x\cos^2 x\right)=\cos^2 x-2x\cos x\sin x$.

(f) Let $u=x^3+1, du=3x^2\ dx.$

When $x=0, u=1;$ when $x=2, u=9.$

$\int_0^2 x^2e^{x^3+1}dx=\frac{1}{3}\int_1^9 e^u du=\frac{1}{3}\left[e^u\right]_1^9=\frac{e^9-e}{3}.$

Q2

(a) $P(1)=2, \therefore 1-a+b=2, \therefore a-b=-1.$

$P(-2)=5, \therefore -8+2a+b=5, \therefore 2a+b=13.$

$3a=12, \therefore a=4.$

$b=a+1=5.$

(b)

(i) $3\sin x+4\cos x=5\sin\left(x+\tan^{-1}\frac{4}{3}\right)$.

(ii) $5\sin\left(x+\tan^{-1}\frac{4}{3}\right)=5$.

$\sin\left(x+\tan^{-1}\frac{4}{3}\right)=1$

$x+\tan^{-1}\frac{4}{3}=\frac{\pi}{2}.$

$x=\frac{\pi}{2}-\tan^{-1}\frac{4}{3}=0.64.$

(c)

(i) $m=\frac{dy/dt}{dx/dt}=\frac{2t}{2}=t.$

$y-t^2=t(x-2t).$

$y=tx-2t^2+t^2=tx-t^2.$

(ii) $y=(2t)x-(2t)^2=2tx-4t^2.$

$y=tx-t^2.$ (1)

$y=2tx-4t^2.$ (2)

$(2)-(1)$ gives

$0=tx-3t^2.$

$x=3t, t\neq 0.$

$y=3t^2-t^2=2t^2.$

$\therefore R\left(3t, 2t^2\right)$

(iii) $x=3t, \therefore t=\frac{x}{3}.$

$y=2t^2=\frac{2x^2}{9}.$

Q3

(a)

(i) The range of e^{2x} is $y>0, \therefore$ The range of $f(x)$ is $y>\frac{3}{4}$

(ii) $f^{-1}: x=\frac{3+e^{2y}}{4}.$

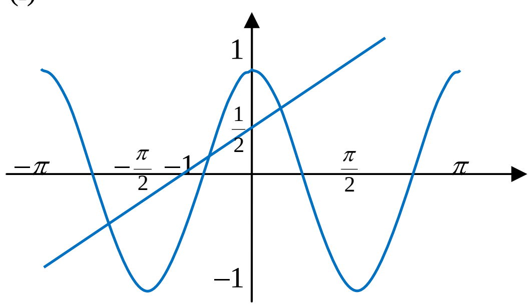
$4x-3=e^{2y}.$

$2y=\ln(4x-3).$

$y=\frac{1}{2}\ln(4x-3).$

(b)

(i)



(ii) Three points of intersection, $\therefore$ Three solutions.

(iii) Let $f(x)=2\cos 2x-x-1.$

$f'(x)=-4\sin 2x-1.$

$x_1=0.4-\frac{2\cos 0.8-0.4-1}{-4\sin 0.8-1}=0.398.$

(c)

(i) RHS $=\frac{1-\left(1-2\sin^2\theta\right)}{1+\left(2\cos^2\theta-1\right)}=\tan^2\theta=\text{LHS}.$

(ii) Let $\theta=\frac{\pi}{8}$.

$$\tan^2\frac{\pi}{8}=\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{\left(\sqrt{2}-1\right)^2}{2-1}$$

$=\left(\sqrt{2}-1\right)^2.$

$\therefore \tan\frac{\pi}{8}=\sqrt{2}-1,$ since $\frac{\pi}{8}$ lies in the 1st quadrant,

$\therefore \tan\frac{\pi}{8}>0.$

Q4

(a)

(i) ${}^5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2=\frac{45}{512}$.

(ii) ${}^5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2+{}^5C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)+\left(\frac{1}{4}\right)^5$

$=\frac{45}{512}+\frac{15}{1024}+\frac{1}{1024}=\frac{53}{512}$.

(iii) $1-\left(\frac{3}{4}\right)^5=\frac{781}{1024}$.

(b)

(i) $f(-x)=\frac{(-x)^4+3(-x)^2}{(-x)^4+3}=\frac{x^4+3x^2}{x^4+3}=f(x)$.

$\therefore f(x)$ is even.

(ii) $y=1$.

(iii) $f'(x)=\frac{(4x^3+6x)(x^4+3)-4x^3(x^4+3x^2)}{(x^4+3)^2}$

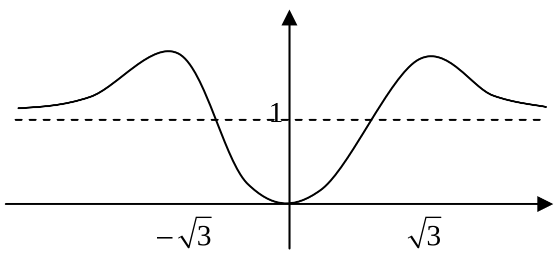
$=\frac{4x^7+12x^3+6x^5+18x-4x^7-12x^5}{(x^4+3)^2}$

$=\frac{-6x^5+12x^3+18x}{(x^4+3)^2}=\frac{-6x(x^4-2x^2-3)}{(x^4+3)^2}$

$=\frac{-6x(x^2-3)(x^2+1)}{(x^4+3)^2}.$

$f'(x)=0$ when $x=0,\pm\sqrt{3}.$

(iv)



(Note: The SPs are (0,0) and $\left(\pm\sqrt{3},\frac{3}{2}\right)$).

Q5

(a)

(i) $\frac{d^2x}{dt^2}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-n^2x.$

$\therefore \frac{1}{2}v^2=-\frac{n^2x^2}{2}+C.$

When $v=0, x=a, \therefore C=\frac{n^2a^2}{2}.$

$\therefore \frac{1}{2}v^2=-\frac{n^2x^2}{2}+\frac{n^2a^2}{2}.$

$\therefore v^2=n^2\left(a^2-x^2\right).$

(ii) Maximum speed occurs when $x = 0$,

$\therefore v^2=n^2a^2, \therefore v=na$.

(iii) Maximum acceleration occurs when $x = a$,

$\therefore a=-n^2x=-n^2a, \therefore$ Max $|a|=n^2a$.

(iv) Let $x=a\sin nt$.

When $x=\frac{a}{2}$,

$\frac{a}{2}=a\sin nt.$

$\sin nt=\frac{1}{2}.$

$nt=\frac{\pi}{6}.$

$t=\frac{\pi}{6n}.$

(b)

(i) The base of the triangle = $2h\tan 60°$.

$\therefore V=10\times\frac{1}{2}\times h\times h\tan 60°=10\sqrt{3}h^2$.

(ii) Area = base of the triangle × 10

= $20\sqrt{3}h$.

(iii) $\frac{dV}{dh}=20\sqrt{3}h.$

$\frac{dh}{dt}=\frac{dh}{dV}\frac{dV}{dt}=\frac{1}{20\sqrt{3}h}\times -k20\sqrt{3}h=-k.$

(iv) By integration, $h=-kt+C$.

When $t=0, h=3, \therefore C=3$

When $t=100, h=2, \therefore 2=-100k+3, \therefore k=0.03.$

$\therefore h=-0.003t+3.$

When $h=1$,

$1=-0.003t+3.$

$t=\frac{2}{0.003}=666.7$ days.

Q6

(a)

(i) When $x_1 = x_2$,

$UT\cos\theta = R - VT\cos\theta.$

$T(U+V)\cos\theta = R.$

$\therefore T = \frac{R}{(U+V)\cos\theta}.$

(ii) When $y_1 = y_2$,

$Ut\sin\theta - \frac{1}{2}gt^2 = h - Vt\sin\theta - \frac{1}{2}gt^2.$

$(U+V)t\sin\theta = h.$

But $h = R\tan\theta, \therefore (U+V)t\sin\theta = R\tan\theta.$

$t = \frac{R\tan\theta}{(U+V)\sin\theta} = \frac{R}{(U+V)\cos\theta}.$

This is the same as the result in (i), $\therefore$ the two particles collide.

(iii) Let $x_1 = \lambda R$ and substitute $t = \frac{R}{(U+V)\cos\theta}$ in the formula $x_1 = Ut\cos\theta$ gives $x_1 = \frac{UR}{(U+V)}$.

$\therefore \lambda R = \frac{UR}{(U+V)}.$

$\therefore \lambda U + \lambda V = U.$

$\lambda V = U(1-\lambda).$

$\therefore V = \left(\frac{1-\lambda}{\lambda}\right)U = \left(\frac{1}{\lambda} - 1\right)U.$

(b)

(i) The GP has the first term $(1+x)^r$, ratio $(1+x)$, and $(n-r+1)$ terms.

$S = \frac{(1+x)^r\left((1+x)^{n-r}-1\right)}{1+x-1} = \frac{(1+x)^r\left((1+x)^{n-r+1}-1\right)}{x}.$

$\therefore (1+x)^r + (1+x)^{r+1} + \ldots + (1+x)^n$

$= \frac{(1+x)^r\left((1+x)^{n-r+1}-1\right)}{x}$

$= \frac{(1+x)^{n+1} - (1+x)^r}{x}.$

The coefficient of x^r in $(1+x)^n$ is $\binom{n}{r}$, $\therefore$ The coefficient of x^r in the LHS is $\binom{r}{r} + \binom{r+1}{r} + \ldots + \binom{n}{r}$.

The coefficient of x^{r+1} in $(1+x)^{n+1}$ is $\binom{n+1}{r+1}$ and the term $(1+x)^r$ does not contain x^{r+1}.

$\therefore$ The coefficient of x^{r+1} in the RHS is $\binom{n+1}{r+1}$.

(ii)

(1) The line $y = x$ passes through the n points along the diagonal, $\therefore$ an interval is formed by choosing any 2 points from the n points on the line. $\therefore \binom{n}{2}$.

(2) The lines that are parallel with the diagonal $y = x$ and lie above it go through $(n-1), (n-2), \ldots (2)$ points so we can form $\binom{n-1}{2}, \binom{n-2}{2}, \ldots \binom{2}{2}$ intervals.

Similarly, the lines that are parallel with the diagonal $y = x$ and lie below it go through $(n-1), (n-2), \ldots (2)$ points so we can also form $\binom{n-1}{2}, \binom{n-2}{2}, \ldots \binom{2}{2}$ intervals.

$\therefore$ Total number of intervals is

$\binom{n}{2} + \binom{n-1}{2} + \binom{n-2}{2} + \ldots + \binom{2}{2} + \binom{n-1}{2} + \binom{n-2}{2} + \ldots + \binom{2}{2},$ (1)

which is the same as

$\binom{2}{2} + \binom{3}{2} + \ldots + \binom{n-1}{2} + \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{3}{2} + \binom{2}{2}$

(iii) Let r = 2, the result in (i) can be rewritten as

$\binom{2}{2} + \binom{3}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}.$

$\therefore$ The result of line (1) becomes

$\binom{n}{2} + \binom{n}{3} + \binom{n}{3}$, which is

$\frac{n!}{2!(n-2)!} + 2\frac{n!}{3!(n-3)!}$

$= \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3}$

$= \frac{n(n-1)}{6}\left(3 + 2(n-2)\right)$

$= \frac{n(n-1)(2n-1)}{6}.$

Q7

(a)

(i) $\frac{d}{dx}(x) = \lim_{h\to 0}\frac{(x+h)-x}{h} = 1.$

(ii) Let $n = 1, \frac{d}{dx}(x) = 1$ (proven above) $= 1x^0$. $\therefore$ True for n = 1.

Assume $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

RTP $\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n$.

$\text{LHS} = \frac{d}{dx}\left(x.x^n\right) = x^n + xnx^{n-1}$

$= x^n + nx^n = (n+1)x^n = \text{RHS}.$

$\therefore$ True for $n + 1$.

$\therefore$ True for all $n \geq 1$.

(b)

(i) Let $\theta = \alpha - \beta$.

$\tan\alpha = \frac{a+h}{x}, \tan\beta = \frac{h}{x}.$

$\tan\theta = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{\frac{a+h}{x} - \frac{h}{x}}{1+\frac{(a+h)h}{x^2}} = \frac{a}{\frac{x^2+(a+h)h}{x}} = \frac{ax}{x^2+(a+h)h}.$

$\therefore \theta = \tan^{-1}\frac{ax}{x^2+h(a+h)}.$

(ii) $\frac{d}{dx}\left(\frac{ax}{x^2+h(a+h)}\right) = \frac{a\left(x^2+h(a+h)\right)-2ax^2}{\left(x^2+h(a+h)\right)^2}$

$= \frac{-ax^2+ah(a+h)}{\left(x^2+h(a+h)\right)^2}.$

$$\frac{d\theta}{dx} = \frac{\frac{-ax^2+ah(a+h)}{\left(x^2+h(a+h)\right)^2}}{1+\left(\frac{ax}{x^2+h(a+h)}\right)^2} = \frac{-ax^2+ah(a+h)}{\left(x^2+h(a+h)\right)^2+a^2x^2}.$$

$\frac{d\theta}{dx} = 0$ when $x^2 = h(a+h)$.

$\therefore x = \sqrt{h(a+h)}.$

This value satisfies $\frac{d\theta}{dx} = 0$ and $x > 0$, θ is minimum when $x = \sqrt{h(a+h)}$.

(c)

(i) $\phi = \theta + \angle SRP$ (in a Δ, the exterior angle equals the sum of the two opposite interior angles).

$\therefore \theta < \phi$.

$\therefore \theta$ is maximum when $\theta = \phi$, which happens when P and T are the same point.

Alternatively, from (b), θ is maximum when $OP^2 = x^2 = h(a+h)$, where O be the point of intersection of PT and QR.

But $OT^2 = OR \times OQ$ (the square of the tangent is equal the product of a secant and its external part).

$\therefore OT^2 = h(a+h)$. (1)

$\therefore OT = OP$.

$\therefore P$ and T are the same point.

(ii) From (1), $OT = \sqrt{h(a+h)}$.